# Beyond Reasoning about Rationality: Evidence of Strategic Reasoning*

Amanda Friedenberg† Terri Kneeland‡

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**Abstract**

While rationality and common belief of rationality (RCBR) is an important benchmark, experiments point to bounds on such reasoning. Yet, the literature is silent on an important question: Do players systematically engage in higher-order strategic reasoning above and beyond reasoning about rationality? We address this question through a novel identification strategy, which separately identifies reasoning about rationality from more general forms of strategic reasoning. We find that at least 60% of subjects engage in forms of strategic reasoning that are not captured by reasoning about rationality. Moreover, the experiment provides insight into how subjects reason when they depart from RCBR.

Game theoretic analyses typically assume players exhibit strategic sophistication. In epistemic game theory, such sophistication is traditionally formalized as rationality and common belief of rationality (RCBR): Players choose a best response given their belief about the play of the game; they believe others do the same; etc. While RCBR is an important theoretical benchmark, lab experiments have pointed to departures from RCBR. See Stahl and Wilson (1995), Costa-Gomes, Crawford, and Broseta (2001), Costa-Gomes and Crawford (2006), Kneeland (2015), and Healy (2024).[1]

Why do we observe such departures from RCBR? The literature has pointed to two types of answers. First, there are difficulties in engaging in interactive reasoning, i.e., reasoning through sentences of the form "I think that you think that I think. . . ." This, in turn, can limit the ability to reason according to RCBR: For Ann to engage in RCBR, she must

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†University of Michigan, amanda@amandafriedenberg.org, http://amandafriedenberg.org

‡University of Calgary, terri.kneeland@ucalgary.ca, http://tkneeland.com

[1]Behavior thought of as reflecting "bounded reasoning" in the context of the level-$k$ and cognitive hierarchy models may, in fact, be consistent with RCBR. See Section 7.A for concrete examples.

specify what she thinks about Bob's play, what she thinks Bob thinks about her play, etc. Second, there are contextual factors that may cause players to depart from RCBR, e.g., historical reasons, culture, etc. For instance, consider two attorneys, Ann and Bob, who have a history of interacting on cases. This past interaction may lead to a situation in which Ann is not prepared to believe that Bob is rational. Or, alternatively, she may believe Bob is rational but may not be prepared to believe that Bob believes she is rational. Importantly, in this story, Ann is capable of engaging in interactive reasoning and, in particular, may still strategize about Bob's behavior. But, she is simply not willing to bet on RCBR. For instance, she may instead assign probability $p$ to a rational Bob believing she is rational and probability $(1-p)$ to a rational Bob believing she adopts some irrational decision rule.

If departures from RCBR are influenced by contextual factors, then players may well engage in forms of strategic reasoning that go beyond reasoning about rationality. This raises the question: Do players systematically engage in strategic reasoning beyond reasoning about rationality? To address the question, it will be useful to clarify the distinction between rational behavior and, what we call, strategic behavior. A player is *rational* if she plays a best response given her subjective belief about how the game is played—put differently, if she maximizes her expected utility given her subjective belief about how the game is played.[2] A player is *strategic* if she has some theory (or method) for how to play the game. One example of such a theory is maximizing subjective expected utility; thus, a player who is rational is also strategic. However, a player may be strategic and irrational; that is, a player may have a decision criterion for playing the game that departs from subjective expected utility. For instance, maximizing the minimum payoff is an example of strategic but irrational behavior. So, a rational player conforms to the "textbook notion of rationality," while a strategic player conforms to a "generalized notion of rationality."

We will distinguish between reasoning about rationality and strategic reasoning. Loosely:

- *Reasoning About Rationality:* Ann has a *rationality bound* of $m$ if she is rational, believes that Bob is rational, believes that Bob believes that she is rational, and so on, up to the statement that includes the word "rational" $m$ times, but no further.

- *Strategic Reasoning:* Ann has a *strategic bound* of $k$ if she is strategic, believes that Bob is strategic, believes that Bob believes that she is strategic and so on, up to the statement that includes the word "strategic" $k$ times, but no further.

Because a rational player is strategic, Ann's strategic bound must be at least as high as her rationality bound. If her strategic bound is strictly higher than her rationality bound, she engages in strategic reasoning beyond reasoning about rationality.

[2]This terminology is consistent with the use in epistemic game theory. See, e.g., Brandenburger, 2007, Dekel and Siniscalchi, 2014, and Battigalli, Friedenberg, and Siniscalchi, 2021.

Is there a gap between the rationality bound and strategic bound? Addressing this question poses a challenge: identifying the strategic bound. In principle, strategic reasoning is a broad concept. For instance, Ann can believe Bob is strategic if she assigns probability $p$ to Bob being rational and probability $(1-p)$ to Bob maximizing his minimum payoff. More broadly, Ann can believe Bob is strategic if she assigns probability $p$ to Bob being rational and probability $(1-p)$ to Bob adopting some other payoff-dependent rule-of-thumb. Etc. These examples can carry over to higher levels, e.g., where Ann believes Bob is strategic and believes Bob believes she is strategic. So, it is not obvious what (if any) observable implications arise from strategic reasoning—at least broadly defined.[3]

Sections 1-2 point out that we can use a particular class of games to identify strategic bounds as distinct from rationality bounds: permuted ring games (Kneeland, 2015). We conduct an experiment based on this identification strategy. The experiment contains two treatments, each based on a four player permuted ring game. The two ring games are equivalent from the perspective of RCBR, but are not equivalent from the perspective of strategic reasoning. In fact, players do systematically behave differently across these games. (See Section 4.2.) The two games illustrate that players do have a gap between their rationality bounds and their strategic bounds. In particular, 60% of subjects have a rationality bound that is strictly lower than their strategic bound. (Section 5.6 argues that the cross-treatment differences suggest this gap may well be higher.) Moreover, on average, subjects with a gap between their rationality and strategic bounds have higher average earnings than their no-gap counterparts. As we will discuss in Section 4.1, this may reflect a deliberate decision to not engage in common belief of rationality.

From the perspective of identifying whether there is a distinction between strategic and rationality bounds, we impose few assumptions on the nature of strategic reasoning. This is an important and deliberate feature of our analysis. A more restrictive model of strategic reasoning would have the benefit of yielding a tighter set of predictions; but, it would come at the cost of imposing auxiliary assumptions about reasoning and beliefs—assumptions that would require separate verification. Moreover, incorrect auxiliary assumptions could give way to false negatives, leading the analyst to conclude there is no strategic reasoning above reasoning about rationality. With this in mind, we seek to understand whether strategic reasoning, broadly construed, plays a role beyond reasoning about rationality. Permuted ring games allow for identification with few auxiliary assumptions. (See Section 2.)

Having established that there is a distinction between strategic and rationality bounds, we would like to better understand the nature of strategic reasoning. As a start, we point to specific assumptions about strategic reasoning that are, on the one hand, strong and, on the

[3]We could address the question narrowly, by one-by-one pitting a specific model of strategic reasoning against reasoning about rationality. We will soon explain why we refrain from taking this approach.

other hand, arguably intuitive. Those assumptions take the following form: If Ann believes Bob is strategic, she assigns probability $p$ to Bob being rational and probability $(1-p)$ to Bob playing according to a specific simple belief-independent heuristic. The "heuristic part" of Ann's beliefs is driven by exactly three rules-of-thumb. (These rules-of-thumb correspond to Bob having a theory that maximizes the maximum payoff, the minimum payoff, or the sum of payoffs.) We refer to this model as the *heuristic beliefs model*. The model provides refined predictions of behavior in our two treatments. Moreover, it suggests cross treatment differences in the distributions of behavior and identified rationality bounds.

Ultimately, the data either refutes or provides evidence in favor of the heuristic beliefs model. We show that 88% of subjects play in accordance with the model. Moreover, we observe the suggested cross-treatment differences. Section 5.7 augments these results, by looking at ring game data from other studies; it points out that the data in those studies are also consistent with the heuristic beliefs model.

Overall, the experimental results point to players engaging in strategic reasoning beyond reasoning about rationality. This implies that the rationality bound is influenced by contextual factors and not determined by limits in the ability to reason interactively. (If the bounds on reasoning about rationality were determined by limits in the ability to engage in interactive reasoning, then there is no room for players to reason strategically beyond how they reason about rationality. This is by definition.)

This conclusion has important implications for understanding behavior in games. Consider a player with a rationality bound of $m$. If that bound were determined by limited ability to engage in interactive reasoning, then our prediction would be *any* $m$-undominated strategy—i.e., any strategy that survives $m$ rounds of deletion of iterated strong dominance. (This follows from standard results, e.g., Tan and da Costa Werlang, 1988.) However, if not, the player may well engage in higher levels of strategic reasoning and that may rule out certain $m$-undominated strategies. This would suggest a refinement of the $m$-undominated strategies. The main results suggest that, broadly speaking, there may be such a relevant refinement.[4] Moreover, in pointing to certain heuristic beliefs that are supported by the data, the results here take a first step toward understanding the nature of that refinement. That is, the results can be seen as an integral step in showing how players strategically reason when they depart from the RCBR benchmark.

**Relationship to the Literature** This paper fits into a new and growing literature—one that attempts to bridge the gap between epistemic game theory and experimental economics. The literature has theoretical (Brandenburger, Danieli, and Friedenberg, 2021; Siniscalchi, 2022) and experimental (Ghosh, Heifetz, and Verbrugge, 2015; Ghosh and Verbrugge, 2018;

[4]The results do not support a refinement based on level-$k$ and cognitive hierarchy models. See Section 6.3.

Healy, 2024; Kneeland, 2015; Li and Schipper, 2020) contributions. Epistemic experiments often take a revealed preference approach. They assume players behave in accordance with some form of rationality (potentially a generalized notion of rationality) and use observed behavior to draw inferences about how players reason. Kneeland (2015) is a prominent example of this approach. The goal of Kneeland was to infer an *exact* level of reasoning about rationality. It showed that this can be achieved in permuted ring games under two key identifying assumptions: subjects are rational and an exclusion restriction is satisfied.[5] We depart from the exclusion restriction and instead only identify the *maximum* level of reasoning about rationality consistent with the data. Our paper and identification strategy point out why the permutation of ring games is a tool that is useful for identifying forms of interactive reasoning beyond reasoning about rationality. In this sense, the paper provides a methodological contribution to the study of epistemic experiments.

Healy (2024) is a complementary contribution. Both this paper and Healy are directly concerned with departures from RCBR. But they use different techniques and address different questions. In particular, Healy fixes a game and elicits both preferences and beliefs; this allows him to address the extent to which players are rational and believe others are rational. By contrast, here we assume rationality and, instead, vary the game in a way that allows us to draw inferences on the extent to which there is strategic reasoning, when there are departures from RCBR. By doing so, it allows us to start a conversation about specific models of strategic reasoning. In this sense, the paper provides an important substantive contribution, one that will help build new models in epistemic game theory.

A second complementary contribution is Sprenger and Zhao (2021). They too are concerned with how simple heuristics can shape beliefs; their focal beliefs model can be reinterpreted as a specific case of the heuristic beliefs model. In Section 7.D, we point to the fact that our data is inconsistent with the "focal representation" of the heuristic beliefs model.

We view the data through the lens of epistemic game theory and the associated identifying assumptions. That said, it is worth stepping back and asking whether the data could, instead, have been generated by an alternate model of decision making. We consider three possibilities: first, noisy decision making (McFadden, 1974; Harless and Camerer, 1994) or measurement error (Gillen, Snowberg, and Yariv, 2019); second, random utility (Thurstone, 1927; Block and Marschak, 1960; Becker, DeGroot, and Marschak, 1963; Gul and Pesendorfer, 2006); third, the level-$k$ (Nagel, 1995; Stahl and Wilson, 1994; Costa-Gomes, Crawford, and Broseta, 2001; Costa-Gomes and Crawford, 2006) or cognitive hierarchy (Camerer, Ho, and Chong, 2004) models. We show that the data could not have been generated by these models alone. To better understand this point, focus on noisy decision making. We provide

[5] While *ex ante* natural, the results in this paper tell us that the exclusion restriction—as conceptualized in Kneeland—was wrong. The Principle of Non-Strategic Reasoning below can be viewed as a weakening of Kneeland's (2015) exclusion restriction.

a comparative static that rules out the possibility that the data is only generated by noise in the decision making process (and, indeed, by these alternate models). Moreover, we show that controlling for noise in the decision making process does not change our qualitative conclusions. See Section 6 and Appendix E.

The questions we address are tied to concepts in epistemic game theory, as are the identifying assumptions. This is not to suggest that the concepts we study are the only models of strategic behavior. Nor does it deny the importance of a large (non-epistemic) literature in behavioral game theory. Rather, we think it is important to understand the extent to which epistemic concepts can explain observed behavior. And, if not, to understand how the data can inform new concepts in epistemic game theory. For this reason, it is important to further bridge a gap between epistemic game theory and experiments.

The paper proceeds as follows. Section 1 introduces the idea behind the identification strategy and the heuristic beliefs model. Section 2 describes the identification strategy. Sections 3-4 describe the experimental design and main results. Section 5 formalizes the heuristic beliefs model and revisits the experimental results. Section 6 discusses whether the data could have been generated by an alternate model of decision making. Section 7 provides discussions important for understanding our results and connections to the literature.

# 1 An Illustrative Example

This section uses an example to illustrate how we separately identify the strategic and rationality bounds. Figure 1.1 describes two games, $G$ and $G_*$. The payoff matrices on the left represent Player 1's payoffs and the payoff matrices on the right represent Player 2's payoffs. Write $(\mathrm{e}, \mathrm{f}_*)$ to denote the choice of action e in $G$ and action $\mathrm{f}_*$ in $G_*$. We refer to such an action profile as a *strategy*. We now point to the important features of these games.

For Player 1 (P1, she), the payoff matrix given by $G_*$ is a relabeling of the payoff matrix given by $G$. That is, there is a permutation $\Pi_1 : \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\} \to \{\mathrm{a}_*, \mathrm{b}_*, \mathrm{c}_*, \mathrm{d}_*\}$ with

$$\Pi_1(\mathrm{a}) = \mathrm{d}_* \qquad \Pi_1(\mathrm{b}) = \mathrm{b}_* \qquad \Pi_1(\mathrm{c}) = \mathrm{a}_* \qquad \Pi_1(\mathrm{d}) = \mathrm{c}_*,$$

so that P1's row $\mathrm{e} \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$ in $G$ corresponds to P1's row $\Pi_1(\mathrm{e}) \in \{\mathrm{a}_*, \mathrm{b}_*, \mathrm{c}_*, \mathrm{d}_*\}$ in $G_*$. Moreover, the strategy $(\mathrm{a}, \Pi_1(\mathrm{a})) = (\mathrm{a}, \mathrm{d}_*)$ is the dominant strategy.

For Player 2 (P2, he), the payoff matrices in $G$ and $G_*$ are the same. P2 does not have a dominant strategy; in fact, each of P2's strategies is undominated. However, P2 has a unique *iteratively undominated* ($IU$) strategy—namely, $(\mathrm{b}, \mathrm{a}_*)$.

**Rational vs. Strategic** To illustrate the relationship between rational and strategic behavior, focus on P1. Suppose that P1 is *rational* in the sense that she maximizes her

P1's Payoffs

| P1 \ P2 | a | b | c | d |
|---|---|---|---|---|
| a | 17 | 15 | 18 | 16 |
| b | 15 | 14 | 15 | 15 |
| c | 6 | 4 | 14 | 8 |
| d | 12 | 2 | 2 | 10 |

P2's Payoffs

| P2 \ P1 | a | b | c | d |
|---|---|---|---|---|
| a | 8 | 14 | 4 | 18 |
| b | 16 | 4 | 2 | 10 |
| c | 15 | 17 | 4 | 4 |
| d | 14 | 6 | 20 | 10 |

(a) Figure $G$

P1's Payoffs

| P1 \ P2 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 6 | 4 | 14 | 8 |
| $b_*$ | 15 | 14 | 15 | 15 |
| $c_*$ | 12 | 2 | 2 | 10 |
| $d_*$ | 17 | 15 | 18 | 16 |

P2's Payoffs

| P2 \ P1 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 8 | 14 | 4 | 18 |
| $b_*$ | 16 | 4 | 2 | 10 |
| $c_*$ | 15 | 17 | 4 | 4 |
| $d_*$ | 14 | 6 | 20 | 10 |

(b) Figure $G_*$

Figure 1.1. A Two-Player Example

subjective expected utility—i.e., she chooses a best response given her subjective belief about how P2 plays the game. Then, she would play the dominant strategy $(\mathrm{a}, \Pi_1(\mathrm{a})) = (\mathrm{a}, \mathrm{d}_*)$. Notice that, if she is rational, then she has a specific theory about how to play the game. This is the sense in which we will say that she is also *strategic*.

At least in principle, P1 may be strategic and irrational. For instance, suppose that P1 instead adopts a rule-of-thumb, in which she chooses an action that could potentially lead to a payoff of 6 provided that such an action exists. She does so even if such an action does not maximize her subjective expected utility. For the purpose of illustration, she adopts such a method for playing the game because 6 is her lucky number. In this case, she would choose the "lucky-6" strategy $(\mathrm{c}, \Pi_1(\mathrm{c})) = (\mathrm{c}, \mathrm{a}_*)$.

This example points to the approach we take more generally. Because P1's payoffs in $G_*$ are a relabeling of her payoffs in $G$, P1's strategic behavior varies systematically between the two games: Any theory that leads P1 to play e in $G$ will lead P1 to play $\Pi_1(\mathrm{e})$ in $G_*$. As a consequence, P1's *strategically optimal behavior* corresponds to the graph of the permutation $\Pi_1$—i.e., to the set

$$\text{Strategic}_1 = \{(\mathrm{a}, \Pi_1(\mathrm{a})), (\mathrm{b}, \Pi_1(\mathrm{b})), (\mathrm{c}, \Pi_1(\mathrm{c})), (\mathrm{d}, \Pi_1(\mathrm{d}))\}. \tag{1}$$

We exploit this fact below.[6]

[6]This paragraph implicitly assumes that either (i) P1's strategically optimal behavior does not depend on her beliefs about P2's play, or (ii) P1 has the same beliefs in $G$ and $G_*$. Our identification strategy will not impose these assumptions exogenously. Instead, they will be satisfied endogenously. See the discussion on page 20 immediately following "Identification (Strategic Bound $k \geq 2$)." (Separately, the assumption of "simple-strategic beliefs" in the heuristic beliefs model will also imply (i)-(ii).)

**Reasoning about Rationality vs. Strategic Reasoning** To illustrate the relationship between reasoning about rationality and strategic reasoning, focus on P2. Throughout the discussion, suppose that P2 is rational. (This will be an important identification assumption. See Section 2.2.) We distinguish between three scenarios.

First, suppose P2 reasons about rationality. By this, we mean that P2 *believes*—i.e., assigns probability 1 to the event—that P1 is rational. In this case, he must assign probability 1 to P1 playing the dominant strategy $(\mathrm{a}, \Pi_1(\mathrm{a})) = (\mathrm{a}, \mathrm{d}_*)$. As P2 is rational, he chooses a best response to this belief; thus, he plays the IU strategy $(\mathrm{b}, \mathrm{a}_*)$. This is a *non-constant strategy* for P2—i.e., a strategy that varies across $G$ and $G_*$.

Second, suppose P2 believes that P1 is strategic. Then, P2 has a *2-strategic belief*, i.e., a belief that assigns probability 1 to P1 choosing a strategy in $\text{Strategic}_1$ (Equation (1)). Notice, P2 may well believe that P1 is rational—after all, P1's dominant strategy is strategic. But, P2 need not. For instance, he may assign probability $4/5$ to P1 being rational and probability $1/5$ to the lucky-6 strategy $(\mathrm{c}, \Pi_1(\mathrm{c})) = (\mathrm{c}, \mathrm{a}_*)$. If P2 holds this 2-strategic belief, his best response is to play $(\mathrm{d}, \mathrm{a}_*)$. Notice, this too is a non-constant strategy for P2, but one that differs from the IU strategy.

Third, suppose P2 believes that P1 is *not strategic*. In this case, he reasons that P1 does not have a theory about how to play the game. As a consequence, he thinks that P1's behavior does not depend on specific parameters of the game—including P1's payoffs. Thus, P2 has the same belief about how P1 plays the game in both $G$ and $G_*$. Since P2's payoffs do not vary across the two games, his best response does not vary. That is, P2's best response is to play a *constant strategy*—i.e., $(\mathrm{a}, \mathrm{a}_*)$, $(\mathrm{b}, \mathrm{b}_*)$, $(\mathrm{c}, \mathrm{c}_*)$, or $(\mathrm{d}, \mathrm{d}_*)$.

**Identification** Notice, if P2 believes P1 is rational, then P2 believes P1 is strategic; but, the converse need not hold. As a consequence, a player's strategic bound must be at least as high as her rationality bound; but, it may be strictly higher.

Does there exist a gap between the strategic and rationality bounds? We seek a conservative estimate of the gap. As such, we seek to identify:

(i) The *maximum* level of reasoning about rationality consistent with observed behavior.

(ii) The *minimum* level of strategic reasoning consistent with observed behavior.

To identify these bounds, we assume that observed behavior is rational, in the sense that it is consistent with a player choosing a best response to her belief. As a consequence, we assume that observed behavior is strategic. So, we do not attempt to distinguish rational behavior from strategic behavior. Instead, our identification focuses on reasoning about rationality vs. strategic reasoning. In light of this, we focus on the observed behavior of P2.

| | Observed Behavior | | | |
|---|---|---|---|---|
| | IU | Non-IU Non-Constant 2-Strategic | Constant | Other |
| Rationality Bound | 2 | 1 | 1 | 1 |
| Strategic Bound | 2 | 2 | 1 | NC |
| Gap | No | Yes | No | NC |

Table 1.1. Identification: Observed P2 Behavior
NC = non-classified

Table 1.1 summarizes the identification. The top row points to the identification of the rationality bound. If we observe P2 play the IU strategy, we identify P2 as having a *rationality bound of 2*. His behavior is consistent with being rational and believing that P1 is rational. Notice, a rational P2 may also play the IU strategy, even if he does not assign probability 1 to P1's rationality. (For instance, it is a best response for P2 to play the IU strategy, if he assigns probability .95 to P1 being rational and probability .05 to P1 playing the lucky-6 strategy.) But, because we seek the maximum level of reasoning about rationality consistent with observed behavior, we identify the rationality bound as 2. On the other hand, if we observe P2 play a non-IU strategy, we identify P2 as having a *rationality bound of 1*. His behavior is consistent with being rational, but inconsistent with both being rational and believing that P1 is rational.

The middle row of Table 1.1 points to the identification of the strategic bound. If we observe P2 play a constant strategy, we identify P2 as having a *strategic bound of 1*. Each constant strategy is consistent with P2 being rational and believing that P1 is *not* strategic. For instance, it is a best response for P2 to play $(c, c_*)$, if he assigns probability 1 to P1 playing $(b, b_*)$. This constant strategy $(c, c_*)$ is also consistent with a rational P2 believing that P1 is strategic. (P1's action b is associated with the same payoff row as P1's action $b_*$ and so $(b, \Pi_1(b)) = (b, b_*)$ is also strategic.) But because we seek the minimum level of strategic reasoning consistent with observed behavior, we identify the strategic bound as 1.

Suppose, instead, we observe P2 play a non-constant strategy. The strategy is inconsistent with a rational P2 believing that P1 is not strategic. As such, we do not identify P2 as having a strategic bound of 1. But, if the strategy is also a best response to a 2-strategic belief, we do identify P2 as having a *strategic bound of* 2. Notice, there are non-constant strategies that are not a best response to a 2-strategic belief. If we were to observe this behavior, we would simply not classify a strategic bound and think that the behavior is driven by something outside of our classification or framework.

The bottom row of Table 1.1 points to the identification of a gap. If P2 plays the IU strategy or a constant strategy, there is no gap identified. If P2 plays a non-IU strategy that is both non-constant and a best response under a 2-strategic belief, then a gap is identified. Note, this is an underestimate of the gap. (See page 21.)

**Comparative Static.** Our identification strategy rests on two assumptions: Observed behavior is a consequence of deliberate choice by a rational P2. Moreover, strategically optimal behavior aligns with the permutation $\Pi_1$. That is, if P2 believes P1 is strategic, P2 must assign probability 1 to the graph of the permutation $\Pi_1$.

This raises the question: Could observed behavior, instead, be generated by an alternate process of decision making? That is, could a different model of decision-making generate our observed data? In light of the literature, there are three natural alternate models: noisy decision-making; random utility; and the level-$k$/cognitive hierarchy models. A comparative static treatment allows us to rule out these alternate models of decision-making.

Refer to Figure 1.2, which is a variant of the original game. There are two changes relative to Figure 1.1. First, in the game $G$, two rows of P1 are swapped: b and d. Second, the permutation of P1 has changed; it is now

$$\Pi_1(\mathrm{a}) = \mathrm{d}_* \qquad \Pi_1(\mathrm{b}) = \mathrm{a}_* \qquad \Pi_1(\mathrm{c}) = \mathrm{b}_* \qquad \Pi_1(\mathrm{d}) = \mathrm{c}_*.$$

Importantly, the dominant strategy of P1 remains $(\mathrm{a}, \Pi_1(\mathrm{a})) = (\mathrm{a}, \mathrm{d}_*)$ and the payoff matrices of P2 have not changed. This implies that P2's IU strategy remains $(\mathrm{b}, \mathrm{a}_*)$.

P1's Payoffs

| | | P2 | | | |
|---|---|---|---|---|---|
| | | a | b | c | d |
| P1 | a | 17 | 15 | 18 | 16 |
| | b | 12 | 2 | 2 | 10 |
| | c | 6 | 4 | 14 | 8 |
| | d | 15 | 14 | 15 | 15 |

P2's Payoffs

| | | P1 | | | |
|---|---|---|---|---|---|
| | | a | b | c | d |
| P2 | a | 8 | 14 | 4 | 18 |
| | b | 16 | 4 | 2 | 10 |
| | c | 15 | 17 | 4 | 4 |
| | d | 14 | 6 | 20 | 10 |

(a) Figure $G$

P1's Payoffs

| | | P2 | | | |
|---|---|---|---|---|---|
| | | $\mathrm{a}_*$ | $\mathrm{b}_*$ | $\mathrm{c}_*$ | $\mathrm{d}_*$ |
| P1 | $\mathrm{a}_*$ | 12 | 2 | 2 | 10 |
| | $\mathrm{b}_*$ | 6 | 4 | 14 | 8 |
| | $\mathrm{c}_*$ | 15 | 14 | 15 | 15 |
| | $\mathrm{d}_*$ | 17 | 15 | 18 | 16 |

P2's Payoffs

| | | P1 | | | |
|---|---|---|---|---|---|
| | | $\mathrm{a}_*$ | $\mathrm{b}_*$ | $\mathrm{c}_*$ | $\mathrm{d}_*$ |
| P2 | $\mathrm{a}_*$ | 8 | 14 | 4 | 18 |
| | $\mathrm{b}_*$ | 16 | 4 | 2 | 10 |
| | $\mathrm{c}_*$ | 15 | 17 | 4 | 4 |
| | $\mathrm{d}_*$ | 14 | 6 | 20 | 10 |

(b) Figure $G_*$

Figure 1.2. A Two-Player Example: Changing the Permutation

Because the IU strategy remains unchanged, Figures 1.1 and 1.2 remain unchanged from the perspective of rationality and belief about rationality. Moreover, the two games are equivalent from the perspective of any process of decision-making that is invariant to P1's non-rational strategic incentives. Thus, if the data were generated by any model of decision-making that is invariant to (the irrational part of the) permutation, the distribution

of play across the two games should be the same.

The key is that each of these alternative models of decision-making is invariant to P1's non-rational strategic incentives. We expand on this in Sections 6. To preview: We would expect the distribution of noise or measurement error to be independent of the permutation. We would expect random utility decision-makers to have the same distribution of Bernoulli utility functions across treatments (and, in fact, across $G$ and $G_*$). We would expect that level-0 behavior is independent of the permutation. Thus, if the data were (entirely) generated by any of these models of decision-making, we would expect to see the same distribution of play across the two treatments.

At the same time, if the data is generated by a rational P2 that believes P1 is strategic, the distribution of play across the two games may well be different. To see why, suppose P2 assigns probability .4 to "P1 is rational" and probability .6 to "P1 chooses the lucky-6 strategy." In the original example, P2 assigns .4:.6 to $(a, d_*) : (c, a_*)$. Here, P2 assigns .4:.6 to $(a, d_*) : (c, b_*)$. In the former case, P2's best response is $(d, b_*)$, while in the latter case it is $(d, a_*)$. (See, also, the discussion in Section 5.4.)

To sum up: In the background, there is a question of whether observed behavior is generated by (i) rational decision-makers who engage in strategic reasoning, or (ii) agents who engage in an alternate model of decision making. We address this question by looking at a treatment effect. If the distribution of play is different across the two treatments, we can rule out these alternate models of decision making. A challenge in drawing this conclusion is that the observed distribution of play may well be different from the true distribution of play. That is, in principle, there may be differences in the observed distributions of play, even if the true distributions of play are identical. Statistical tests will allow us to rule out the possibility that the true distributions of play are, indeed, identical.

Sections 4.2 and 6 will expand on this argument. Moreover, there, we provide additional auxiliary evidence that the data was not generated by these alternate models.

**Heuristic Beliefs Model** Suppose P2 does not believe that "P1 is rational," but does believe that "P1 is strategic." In this case, P2 believes that P1 has a theory for how to play the game. In principle, P2's model of P1's theory may be complex. Thus far, we have not taken a stance on what it might be. At the same time, it seems natural that P2's model is shaped by simple heuristics or rules-of-thumb that P1 can adopt.

Let's take a concrete example of one such a heuristic: P1 seeks to maximize the minimum payoff that she can receive. If P2 reasons that this is the *only* criterion that determines P1's strategic belief, then P2 would assign probability 1 to P1 choosing the strategy that is best according to that criterion—i.e., to P1 choosing the dominant strategy $(a, \Pi_1(a))$. If, instead, P2 thinks that this is only one criterion that can shape P1's beliefs, then P2 may

not be prepared to assign probability 1 to the dominant strategy. However, if this heuristic is prominent in P2's model, then he should reason that strategies associated with higher minimum payoffs are more likely. Thus, in Figure 1.1, P2's belief should satisfy

$$\Pr_2(\mathrm{a}, \Pi_1(\mathrm{a})) \geq \Pr_2(\mathrm{b}, \Pi_1(\mathrm{b})) \geq \Pr_2(\mathrm{c}, \Pi_1(\mathrm{c})) \geq \Pr_2(\mathrm{d}, \Pi_1(\mathrm{d})), \tag{2}$$

since $(\mathrm{a}, \Pi_1(\mathrm{a}))$ has a minimum payoff of 15, $(\mathrm{b}, \Pi_1(\mathrm{b}))$ has a minimum payoff of 14, etc.

Of course, maximizing the minimum payoff is only one of many possible heuristics. This paper focuses on a class of three heuristics. The heuristics are based on whether P1 seeks to maximize the maximum, minimum, or sum of payoffs that she can receive. Our goal is to understand whether the class, as whole, is prominent in forming P2's strategic beliefs. That is, the goal is to either falsify or provide evidence in favor of the hypothesis that P2's beliefs are shaped by this class of heuristics. As such, the experiments are designed so that P1 ranks strategies the same way, irrespective of which heuristic in the class is relevant. That is, $(\mathrm{e}, \Pi_1(\mathrm{e}))$ has a higher maximum payoff than $(\mathrm{f}, \Pi_1(\mathrm{f}))$ if and only if $(\mathrm{e}, \Pi_1(\mathrm{e}))$ has a higher minimum payoff and a higher payoff sum than $(\mathrm{f}, \Pi_1(\mathrm{f}))$. In this case, we say that $(\mathrm{e}, \Pi_1(\mathrm{e}))$ *heuristically dominates* $(\mathrm{f}, \Pi_1(\mathrm{f}))$.

**Observable Implications of the Heuristic Beliefs Model** We pointed out that, if a rational P2 believes that P1 is strategic, P2's behavior may vary across Figures 1.1 and 1.2. When P2 holds heuristic beliefs, we can provide a tighter prediction on P2's behavior. To illustrate, it will be useful to understand how the differences between Figures 1.1 and 1.2 impact the nature of heuristic beliefs.

Recall, there are two differences between Figures 1.1 and 1.2. First, in the game $G$, two of P1's rows are flipped: b and d. Second, the permutation $\Pi_1$ changes. Flipping the rows in $G$ impacts how actions in $G$ are ranked according to the heuristics. In Figure 1.1, b heuristically dominates c which heuristically dominates d; in Figure 1.2, d heuristically dominates c which heuristically dominates b. Changing the permutation changes which actions in $G_*$ correspond to the actions b, c, and d. Taken together, a player who holds heuristic beliefs has beliefs that satisfy

$$\Pr_2(\mathrm{a}, \mathrm{d}_*) \geq \Pr_2(\mathrm{b}, \mathrm{b}_*) \geq \Pr_2(\mathrm{c}, \mathrm{a}_*) \geq \Pr_2(\mathrm{d}, \mathrm{c}_*) \tag{3}$$

in Figure 1.1, but

$$\Pr_2(\mathrm{a}, \mathrm{d}_*) \geq \Pr_2(\mathrm{d}, \mathrm{c}_*) \geq \Pr_2(\mathrm{c}, \mathrm{b}_*) \geq \Pr_2(\mathrm{b}, \mathrm{a}_*) \tag{4}$$

in Figure 1.2. These different requirements of heuristic beliefs impact which strategies are vs. are not a best response.

The payoffs in Figures 1.1 and 1.2 were chosen to satisfy the following requirement: Even if P2 forms his heuristic belief systematically—that is, he assign the same probability to

the heuristic-best vs. second-best, etc—P2's best response may differ across the two games. (See Figures 5.1 and D.1.) Thus, there are certain distributions of play that are consistent with the heuristic beliefs model in Figure 1.1 but not Figure 1.2, and visa versa.

**Differences in Identified Rationality Bounds** There is a second observable implication of the Heuristic Belief model: We would expect to see more subjects play IU in Figure 1.2 than in Figure 1.1. To understand why, suppose that P2 holds a heuristic belief. Then, he will assign some probability $p > 0$ to P1 choosing the heuristic-best strategy; P1's heuristic-best strategy is P1's dominant strategy. (Thus, we can view P2 as holding a belief that assigns probability $p$ to P1 being rational and probability $(1 - p)$ to P1 playing an invariant but irrational strategy.) Certainly, if $p = 1$, P2's IU strategy is a best response to such a belief. However, it will also be a best response if $p$ is less than 1 but "high." Consider the minimum probability $p$ that P2 can assign to the dominant strategy of P1 and have the IU strategy as a best response. This probability is smaller in Figure 1.2 than in Figure 1.1. (This feature is by design—that is, the payoffs in Figures 1.1 and 1.2 were chosen so that this property holds.) So, for instance, a rational P2 that always assigns probability .85 to P1's dominant strategy would not play IU in Figure 1.1, but must play IU in Figure 1.2.

**Identified Gap** The differences in the identified rationality bounds have consequences for the identification of a gap between the rationality and strategic bounds. To see this, suppose that players' *actual* rationality and strategic bounds do not vary across treatments. With heuristic beliefs, we would expect Figure 1.1 to be associated with a lower *identified* rationality bound relative to Figure 1.2. So the identified gap will be larger in Figure 1.1 relative to Figure 1.2.

Figures 1.1 vs. 1.2 serve different roles. Both games allow us to identify a gap. However, the identified gap is a lower bound on the actual gap. We would expect Figure 1.2 to underestimate the gap relative to Figure 1.1. For this reason, we think of Figure 1.1 as better suited for identifying the gap. It will be part of, what we will call, an *Identification* treatment. While Figure 1.2 does allow us to identify a gap, its central role is to allow for the comparative static: It allows us to address whether the observed behavior could have been generated by an alternate model of decision making and to investigate the heuristic beliefs model. So, it will be part of, what we will call, a *Comparative Static* treatment.

## 2 Identifying the Gap

This section lays out key assumptions behind our identification strategy in the context of a four-player game. There are two benefits of a four-player over a two-player game. First,

P1's Payoffs

| P1 \ P4 | a | b | c | d |
|---|---|---|---|---|
| a | 17 | 15 | 18 | 16 |
| b | 15 | 14 | 15 | 15 |
| c | 6 | 4 | 14 | 8 |
| d | 12 | 2 | 2 | 10 |

P2's Payoffs

| P2 \ P1 | a | b | c | d |
|---|---|---|---|---|
| a | 8 | 14 | 4 | 18 |
| b | 16 | 4 | 2 | 10 |
| c | 15 | 17 | 4 | 4 |
| d | 14 | 6 | 20 | 10 |

P3's Payoffs

| P3 \ P2 | a | b | c | d |
|---|---|---|---|---|
| a | 12 | 14 | 7 | 20 |
| b | 18 | 4 | 7 | 14 |
| c | 8 | 16 | 2 | 6 |
| d | 2 | 15 | 17 | 8 |

P4's Payoffs

| P4 \ P3 | a | b | c | d |
|---|---|---|---|---|
| a | 6 | 8 | 16 | 2 |
| b | 20 | 12 | 12 | 6 |
| c | 14 | 17 | 4 | 6 |
| d | 8 | 2 | 15 | 18 |

(a) Figure $G$

P1's Payoffs

| P1 \ P4 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 6 | 4 | 14 | 8 |
| $b_*$ | 15 | 14 | 15 | 15 |
| $c_*$ | 12 | 2 | 2 | 10 |
| $d_*$ | 17 | 15 | 18 | 16 |

P2's Payoffs

| P2 \ P1 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 8 | 14 | 4 | 18 |
| $b_*$ | 16 | 4 | 2 | 10 |
| $c_*$ | 15 | 17 | 4 | 4 |
| $d_*$ | 14 | 6 | 20 | 10 |

P3's Payoffs

| P3 \ P2 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 12 | 14 | 7 | 20 |
| $b_*$ | 18 | 4 | 7 | 14 |
| $c_*$ | 8 | 16 | 2 | 6 |
| $d_*$ | 2 | 15 | 17 | 8 |

P4's Payoffs

| P4 \ P3 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 6 | 8 | 16 | 2 |
| $b_*$ | 20 | 12 | 12 | 6 |
| $c_*$ | 14 | 17 | 4 | 6 |
| $d_*$ | 8 | 2 | 15 | 18 |

(b) Figure $G_*$

Figure 2.1. Identification Game

it allows us to draw inferences up to four levels of reasoning. Second, it allows us to use behavior across player roles to tighten the identification.

Figure 2.1 describes two games, $G$ and $G_*$. Each of the games has a *ring structure* (Kneeland, 2015; Brocas, Carrillo, and Sachdeva, 2018): Player $i$'s (P$i$'s) payoffs depend only on the behavior of Player $(i-1)$ (P$(i-1)$).[7] (We adopt the convention that P0 $\equiv$ P4.) There are three important features of the games. First, P1's and P2's payoff matrices are as in Figure 1.1. So, for P1, the payoff matrix in $G_*$ is a relabeling of the payoff matrix in $G$ and P2 has the same payoff matrix across the two games. P3 and P4 also have the same matrices across the two games. Second, while P1 has a dominant strategy, all strategies are undominated for P2, P3, and P4. Third, the games are dominance solvable.

Figure 2.2 also describes two games with a ring structure. The only difference between Figure 2.1 and 2.2 is P1's payoff matrix. Whereas P1's payoffs in Figure 2.1 correspond to Figure 1.1, P1's payoffs in Figure 2.2 correspond to Figure 1.2. Thus, round-for-round, iterated dominance is the same across Figures 2.1 and 2.2. Given the connection to Figures 1.1-1.2, we refer to Figure 2.1 as the *Identification Game* and Figure 2.2 as the *Comparative Static Game*. Each subject will play one of these games, not both. For the purpose of describing how the rationality and strategic bounds are identified, we will focus our discussion on the Identification Game (Figure 2.1).

Each subject plays both $G$ and $G_*$ in each player role. As such, an observation consist of a subject's behavior across eight games—that is, an observation is a profile $x =$

[7] Brocas, Carrillo, and Sachdeva introduced a ring structure, independently of Kneeland. They do not use permuted ring games, which is a key feature of our design and Kneeland's.

P1's Payoffs

| P1 \ P4 | a | b | c | d |
|---|---|---|---|---|
| a | 17 | 15 | 18 | 16 |
| b | 12 | 2 | 2 | 10 |
| c | 6 | 4 | 14 | 8 |
| d | 15 | 14 | 15 | 15 |

P2's Payoffs

| P2 \ P1 | a | b | c | d |
|---|---|---|---|---|
| a | 8 | 14 | 4 | 18 |
| b | 16 | 4 | 2 | 10 |
| c | 15 | 17 | 4 | 4 |
| d | 14 | 6 | 20 | 10 |

P3's Payoffs

| P3 \ P2 | a | b | c | d |
|---|---|---|---|---|
| a | 12 | 14 | 7 | 20 |
| b | 18 | 4 | 7 | 14 |
| c | 8 | 16 | 2 | 6 |
| d | 2 | 15 | 17 | 8 |

P4's Payoffs

| P4 \ P3 | a | b | c | d |
|---|---|---|---|---|
| a | 6 | 8 | 16 | 2 |
| b | 20 | 12 | 12 | 6 |
| c | 14 | 17 | 4 | 6 |
| d | 8 | 2 | 15 | 18 |

(a) Figure $G$

P1's Payoffs

| P1 \ P4 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 12 | 2 | 2 | 10 |
| $b_*$ | 6 | 4 | 14 | 8 |
| $c_*$ | 15 | 14 | 15 | 15 |
| $d_*$ | 17 | 15 | 18 | 16 |

P2's Payoffs

| P2 \ P1 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 8 | 14 | 4 | 18 |
| $b_*$ | 16 | 4 | 2 | 10 |
| $c_*$ | 15 | 17 | 4 | 4 |
| $d_*$ | 14 | 6 | 20 | 10 |

P3's Payoffs

| P3 \ P2 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 12 | 14 | 7 | 20 |
| $b_*$ | 18 | 4 | 7 | 14 |
| $c_*$ | 8 | 16 | 2 | 6 |
| $d_*$ | 2 | 15 | 17 | 8 |

P4's Payoffs

| P4 \ P3 | $a_*$ | $b_*$ | $c_*$ | $d_*$ |
|---|---|---|---|---|
| $a_*$ | 6 | 8 | 16 | 2 |
| $b_*$ | 20 | 12 | 12 | 6 |
| $c_*$ | 14 | 17 | 4 | 6 |
| $d_*$ | 8 | 2 | 15 | 18 |

(b) Figure $G_*$

Figure 2.2. Comparative Static Game

$(x(1), x(2), x(3), x(4))$, where each $x(i) \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\} \times \{\mathrm{a}_*, \mathrm{b}_*, \mathrm{c}_*, \mathrm{d}_*\}$ indicates the subject's behavior in the role of P$i$ across both $G$ and $G_*$. We assume that each subject is rational (and, so, strategic). Therefore, we can use the subjects' behavior across both games and player roles to provide a lower bound on strategic reasoning and an upper bound on reasoning about rationality. This provides us with a conservative estimate (i.e., an underestimate) of the gap between the strategic and rationality bounds.

The next subsections will elaborate on the identification strategy. But, Section 1 provided the basic insight: If we identify a subject as having a rationality bound of $m$, the subject plays IU in the role of P$i$=P1,...,P$m$—but not in the role of P$(m+1)$. If we identify a subject as having a strategic bound of $m$, the subject plays a non-constant strategy profile in the role of P$m$ and a constant strategy profile in the role of P$i$ for each $i > m$.

## 2.1 Identifying the Rationality Bounds

Say a subject is *1-rational* if, in the role of each P$i$, she plays a best response given a belief about P$(i-1)$'s play of the game. Say a subject is *m-rational* if, in the role of each P$i$, she plays a best response given a belief that assigns probability 1 to the event that P$(i-1)$ is $(m-1)$-*rational*. Note, a subject is $m$-rational if and only if she satisfies *rationality and* $(m-1)^{th}$*-order belief of rationality*. (See, e.g., Remark 5.1 in Battigalli, Friedenberg, and Siniscalchi, 2021 for this standard result.) Say the subject has a *rationality bound of* $m$ if she is $m$-rational but not $(m+1)$-rational.

There is a tight connection between $m$-rationality and the IU strategies: If a subject is $m$-rational then, in the role of each P$i$, she plays a strategy that survives $m$ rounds

of iterated dominance. Conversely, if a subject plays a strategy that survives $m$ rounds of iterated dominance, then her behavior is consistent with $m$-rationality. (See, e.g., Tan and da Costa Werlang (1988), Battigalli and Siniscalchi (2002), among others.) An observation $x = (x(1), x(2), x(3), x(4))$ survives $m$ rounds of iterated dominance if and only if $(x(1), \ldots, x(m))$ is IU. As such:

**Identification** (Rationality Bound)**.** *Identify* $x = (x(1), x(2), x(3), x(4))$ *as having a* rationality bound of $m$ *if*

*(i)* $(x(1), \ldots, x(m))$ *is IU; and*

*(ii) if* $m = 1, 2, 3$*, then* $x(m+1)$ *does not survive IU.*

An observation is identified as having a rationality bound of $m$ if the observed behavior is consistent with $m$-rationality and, when $m < 4$, the observed behavior is inconsistent with $(m+1)$-rationality. If a subject actually has a rationality bound of $m \leq 4$, then her behavior generates an observation with an identified rationality bound of $n \geq m$. But, if a subject is actually $m$-rational for $m > 4$, then her behavior generates an observation with an identified rationality bound of 4.

Importantly, the identification strategy makes use of the subject's behavior across all player roles. For instance, suppose we observe $x = (x(1), x(2), x(3), x(4))$, where $x(4)$ is IU and $x(2)$ is not IU. If we identified the rationality bound by only looking at observed behavior in a single player role, then we might use the fact that $x(4)$ is IU to incorrectly conclude that the subject's rationality bound is 4. However, the behavior in the role of P2 (i.e., $x(2)$) does not survive two rounds of iterated dominance. As such, the observation $x$ is inconsistent with a rational subject who assigns probability 1 to the event that "P1 is rational." As a consequence, the subject's behavior is inconsistent with 4-rationality. Thus, we identify this observation as having a rationality bound of 1.

## 2.2 Identifying the Strategic Bounds

To identify a subject's reasoning, we will assume that she is rational—not simply strategic. Moreover, we assume that no subject's behavior (in any player role) is a result of indifference.[8] This implies that each subject chooses amongst pure strategies. In addition, we assume that each subject believes that others choose amongst pure strategies.[9]

[8] The data in this paper supports this assumption. This can be seen by applying the argument in Kneeland's (2015) footnote 20 to our dataset. Kneeland's data also supports this assumption. (Footnote 20 discusses rationalizable strategies, but the same argument applies to all strategies.)

[9] This will effectively follow from assumptions about strategic reasoning that we impose below. As in the epistemic game theory literature, this does *not* imply that players have degenerate beliefs.

With this in mind, we think of the subject as having a belief in each player role P$i$ about P$(i-1)$'s behavior across $G$ and $G_*$. Write $\Pr_i$ for P$i$'s distribution on $\{\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}\} \times \{\mathrm{a}_*,\mathrm{b}_*,\mathrm{c}_*,\mathrm{d}_*\}$. So, $\Pr_i(\mathrm{e},\mathrm{f}_*)$ is the probability that P$i$ assigns to P$(i-1)$ playing the strategy $(\mathrm{e},\mathrm{f}_*)$. We must specify what it means for the subject to believe that the other players are not strategic vs. are strategic. For that, we introduce two identification assumptions.

**Identifying Non-Strategic Reasoning** Consider the case in which a subject believes that others are not strategic. Section 1 introduced the basic idea: If P$i$ believes that P$(i-1)$ is not strategic, she reasons that P$(i-1)$ does not have a theory about how to play the game. As a consequence, she believes that P$(i-1)$'s behavior does not depend on specific parameters of the game—including P$(i-1)$'s payoffs. Thus, P$i$ has the same belief about how P$(i-1)$ plays the game in both $G$ and $G_*$.

This is the approach we take more generally. If a subject believes that others are not strategic, then she reasons that their behavior does not depend on the details of the game. This implies that, within a given player role, she reasons that the behavior of others does not depend on whether $G$ versus $G_*$ is played. But, within a given a game, it also implies that she reasons that the behavior of others does not depend on the player role.[10]

Say a subject believes others are not strategic if she satisfies:

> **Principle of Non-Strategic Reasoning:** The subject has the same belief Pr in each player role, i.e, $\Pr_i = \Pr$ for each $i = 1,2,3,4$. Moreover, this belief satisfies $\Pr(\mathrm{a},\mathrm{a}_*) + \Pr(\mathrm{b},\mathrm{b}_*) + \Pr(\mathrm{c},\mathrm{c}_*) + \Pr(\mathrm{d},\mathrm{d}_*) = 1$.

Call a belief for P$i$, $\Pr_i$, a *constant belief* if $\Pr_i(\mathrm{a},\mathrm{a}_*)+\Pr_i(\mathrm{b},\mathrm{b}_*)+\Pr_i(\mathrm{c},\mathrm{c}_*)+\Pr_i(\mathrm{d},\mathrm{d}_*) = 1$. The Principle of Non-Strategic Reasoning says that the subject has the same constant belief in each player role.

**Identifying Strategic Reasoning** Consider the case in which a subject believes that others are strategic. Section 1 introduced the basic idea: If P$i$ believes that P$(i-1)$ is strategic, she reasons that P$(i-1)$'s decisions are determined by his payoff matrix and potentially his beliefs about play. For instance, P$i$ may believe that P$(i-1)$ adopts a rule-of-thumb in which he always chooses a strategy that generates the highest maximum payoff—or, alternatively, that P$(i-1)$ adopts a rule-of-thumb that generates the highest minimum or sum of payoffs. Or, alternatively, P$i$ may believe that P$(i-1)$ adopts a rule-of-thumb whereby P$(i-1)$, first, chooses a strategy that could lead to a payoff of 6 if such an action exists and, second, if not, plays a best response given his subjective belief about the play of the game. Or, alternatively, P$i$ may believe that P$(i-1)$ is rational. In each of

[10] This is a reasonable assumption in the experiment, where subjects do not observe the identity of others.

these cases P$i$ believes that P$(i-1)$ has some theory about how to play the game; in the latter two examples, she believes that P$(i-1)$'s theory depends on his beliefs.

More generally, when we think of a subject that believes the other players are strategic, we think of a subject that believes that other players choose a strategically optimal strategy. To understand which behavior is strategically optimal, notice the following: For each player P$i$, there is a permutation $\Pi_i$ of $i$'s actions from $G$ to $G_*$ that preserves P$i$'s payoff matrix. P1's permutation was described on p. 6. For P$i$=P2,P3,P4, the permutation is constant—i.e., mapping each $\Pi_i(\mathrm{e}) = \mathrm{e}_*$. If a strategic P$i$ adopts a rule-of-thumb that does not depend on her belief about P$(i-1)$'s behavior, then her rationale for choosing e in $G$ would also serve as a rationale for playing the permuted $\Pi_i(\mathrm{e})$ in $G_*$. The same conclusion holds for any strategic P$i$ if she has the same beliefs about P$(i-1)$'s behavior across $G$ and $G_*$ (i.e., if the probability she assigns to e in $G$ is the same as the probability she assigns to $\Pi_{(i-1)}(\mathrm{e})$ in $G_*$). Put differently, in these cases, P$i$'s strategic optimality is invariant to the permutation of payoff-equivalent action labels.

With this in mind, let

$$\mathrm{Inv}_i = \{(\mathrm{a}, \Pi_i(\mathrm{a})), (\mathrm{b}, \Pi_i(\mathrm{b})), (\mathrm{c}, \Pi_i(\mathrm{c})), (\mathrm{d}, \Pi_i(\mathrm{d}))\}$$

be the set of $i$'s *invariant strategies*. Say **P$i$ is strategic** if, whenever P$i$ has a constant belief, she plays an invariant strategy, i.e., $(\mathrm{e}, \Pi_i(\mathrm{e})) \in \mathrm{Inv}_i$. To better understand this definition, suppose that P$i$ is strategic. If P$i$ has a non-constant belief, we cannot draw conclusions about her behavior. However, if she has a constant belief, any of her invariant strategies is strategically optimal. Moreover, in that case, the invariant strategies are the only strategies that are strategically optimal, provided P$i$ is not indifferent.

Say a subject believes that others are strategic if, in each player role P$i$, she satisfies:

> **Principle of Strategic Reasoning:** If P$i$ believes that P$(i-1)$ has a constant belief and $\Pr_i(\mathrm{e}, \mathrm{f}_*) > 0$, then $(\mathrm{e}, \mathrm{f}_*) \in \mathrm{Inv}_{i-1}$.

The Principle of Strategic Reasoning captures the idea that P$i$ believes P$(i-1)$ chooses a strategically optimal strategy and is not indifferent between any such strategies. Thus, if P$i$ believes that "P$(i-1)$ has the same beliefs about P$(i-2)$'s behavior across $G$ and $G_*$," then the principle requires P$i$ to believe that P$(i-1)$ plays an invariant strategy.

The Principle only applies when P$i$ believes that P$(i-1)$ has the same beliefs across $G$ and $G_*$. This is deliberate: If P$(i-1)$ has different beliefs across the two games, then even a rational P$(i-1)$ need not play an invariant strategy. In the identification strategy, the Principle will only have bite when P$i$ believes that P$(i-1)$ has a constant belief. (This case will arise endogenously based on a subject's strategic bound; see page 20.)

Note, the Principle implicitly requires that P$i$ has the same belief about the nature of P$(i-1)$'s strategic optimality criterion across $G$ and $G_*$. For instance, P2 cannot believe

"P1 is rational in $G$ and chooses the lucky-6 strategy in $G_*$." If P2 had such a belief, he would not believe P1 plays an invariant strategy (even if he believes P1 has a constant belief). As a second example, suppose P$i$ believes "P$(i-1)$ is rational and breaks any indifferences she may have by using a particular rule-of-thumb." The Principle implicitly assumes that the particular rule-of-thumb does not vary across $G$ and $G_*$. If it did, then P$i$ may not believe P$(i-1)$ chooses an invariant strategy.

**Strategic Bounds** Say a subject is *1-strategic* if, in each player role, she is strategic. Say a subject is *2-strategic* if she is *1-strategic* and, in each player role, she satisfies the Principle of Strategic Reasoning. Inductively, say a subject is *$k$-strategic* if she is $(k-1)$-strategic and she believes (i.e., assigns probability 1 to the event) that other subjects are $(k-1)$-strategic. A subject has a *strategic bound of 1* if she is 1-strategic and satisfies the Principle of Non-Strategic Reasoning. Inductively, a subject has a *strategic bound of $k$* if she is $k$-strategic and believes that other players have a strategic bound of $(k-1)$.

We inductively identify the strategic bounds. Proposition A.1 establishes that this identification corresponds to the minimal strategic bound consistent with observed behavior.[11]

**Identification** (Strategic Bound 1)**.** *Identify* $(x(1), x(2), x(3), x(4))$ *as having a* strategic bound of 1 *if there exists a constant belief* Pr *on* $\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\} \times \{\mathrm{a}_*, \mathrm{b}_*, \mathrm{c}_*, \mathrm{d}_*\}$ *so that each* $x(i)$ *is a unique best response under* Pr*.*

If we identify an observation $x = (x(1), x(2), x(3), x(4))$ as having a strategic bound of 1, then each of $x(1), x(2), x(3)$, and $x(4)$ is a best response under the same constant belief. In the role of P1, the observed behavior $x(1)$ must be non-constant: The dominant strategy $(\mathrm{a}, \mathrm{d}_*)$ is the only strategy that can be a best response to any belief. In the role of P$i$=P2,P3,P4, the observed behavior $x(i)$ must be constant: P$i$=P2,P3,P4 has the same payoff matrix in $G$ and $G_*$ and so only a constant strategy can be a unique best response to a constant belief. (The uniqueness requirement reflects the assumption that no behavior is an artifact of indifference.) Thus, $x \in \{(\mathrm{a}, \mathrm{d}_*)\} \times \{(\mathrm{a}, \mathrm{a}_*), (\mathrm{b}, \mathrm{b}_*), (\mathrm{c}, \mathrm{c}_*), (\mathrm{d}, \mathrm{d}_*)\}^3$. Table C.1 details the observations identified with a strategic bound of 1.

Suppose that we have identified strategic bounds of $1, \ldots, k-1$, where $4 > k-1$. We now identify a strategic bound of $k$.

**Identification** (Strategic Bound $k \geq 2$)**.** *Identify* $(x(1), x(2), x(3), x(4))$ *as having a* strategic bound of $k \geq 2$ *if it has not been identified as having a strategic bound of $j < k$ and the following hold:*

*(i)* $x(1) = (\mathrm{a}, \mathrm{d}_*)$*;*

[11] The formal statements that justify the identified strategic bounds can be found in Appendix A, along with the associated proofs.

*(ii)* $x(2)$ *is a unique best response under a* 2*-strategic belief; and*

*(iii) for each* $j$ *with* $4 \geq j > k$, $x(j)$ *is constant.*

Tables C.2–C.6 detail the observations identified with a strategic bound of $k \geq 2$.

To better understand the identification, focus on a subject who has a strategic bound of 2. This is a subject who, in each player role P$i$ believes that "P$(i-1)$ is strategic and believes P$(i-2)$ is non-strategic." (This uses the fact that she believes that others are strategic and have a strategic bound of 1.) Recall, if P$(i-1)$ believes that "P$(i-2)$ is non-strategic," then P$(i-1)$ has the same belief about P$(i-2)$'s behavior across $G$ and $G_*$. Thus, the Principle of Strategic Reasoning says that, in the role of P$i$, this subject must believe that P$(i-1)$'s behavior is invariant to permuting equivalent action labels. That is, P$i$ assigns probability 1 to $\text{Inv}_{i-1}$. For P$i$=P2, this is a 2-strategic belief and so $x(2)$ must be a unique best response under a 2-strategic belief. For P$i$ = P3, P4, this is a constant belief. Since $x(i) \in \{x(3), x(4)\}$ must be a unique best response and P$i$'s matrix is the same across $G$ and $G_*$, it follows that $x(i) \in \{x(3), x(4)\}$ is constant.

Note, there are observations in $\{(\text{a}, \text{d}_*)\} \times \{(\text{a}, \text{a}_*), (\text{b}, \text{b}_*), (\text{c}, \text{c}_*), (\text{d}, \text{d}_*)\}^3$ that cannot be identified as having a strategic bound of 1; these are observations $(x(1), x(2), x(3), x(4))$ for which there is no single constant belief Pr where each $x(i)$ is a unique best response under Pr. (See Table C.1.) Those observations are constant in player role P$i$=P2,P3,P4. They can be identified with a strategic bound of 2—but only if their behavior in the role of P2 is uniquely optimal under a 2-strategic belief. More generally, an observation is identified as having a strategic bound of 2 if and only if (i) $x(1)$ is the dominant strategy; (ii) $x(2)$ is a unique best response under a 2-strategic belief; and (iii) $x(3), x(4)$ are constant. Many of these observations have $x(2)$ non-constant.

## 2.3 Gap Between the Rationality and Strategic Bounds

The identified rationality bound must be less than the identified strategic bound. (See Lemma A.6.) We seek to identify whether there is a gap between the identified rationality and identified strategic bounds.

**Identification** (Gap versus No Gap)**.**

*(i) Identify* $(x(1), x(2), x(3), x(4))$ *as having a* gap *if the observation is identified with a rationality bound* $m \geq 1$ *and a strategic bound of* $k > m$.

*(ii) Identify* $(x(1), x(2), x(3), x(4))$ *as having* no gap *if the observation is identified with a rationality bound* $m \geq 1$ *and a strategic bound of* $k = m$.

Lemma A.7 shows, if the observation has an identified gap, then the actual rationality bound must be strictly less than the actual strategic bound. If an observation is identified as having a strategic bound of 1 or a rationality bound of 4, then the observation is necessarily identified as having no gap. Even if the observation is identified as no gap, the behavior may still have been generated by a rational subject whose rationality bound is strictly lower than her strategic bound. (The identified rationality bound may be strictly higher than the subject's rationality bound; the identified strategic bound may be strictly lower than the subject's strategic bound.) That is, the identified gap is a conservative estimate of the gap.

## 3 Experimental Design

The experiment was conducted online using the ELFE subject pool at UCL.[12] We used ORSEE (Greiner, 2015) for recruitment and collected data from 295 undergraduate subjects. The experimental program was written in oTree (Chen, Schonger, and Wickens, 2016). Appendix D provides screenshots of the instructions and an example game.

Each subject was assigned to one of two treatments: the Identification (IDENT) Treatment or the Comparative Static (CS) Treatment. IDENT (resp. CS) corresponds to Figure 2.1 (resp. Figure 2.2). Within the assigned treatment, each subject played $G$ and $G_*$ in each of the four player roles. (So, each subject played eight games.) The order of the games was random. Subjects were required to spend at least 90 seconds on each of the games. After making their choices in all games, subjects were given the opportunity to revise their choices. There was no feedback throughout the play of the games and the revision.

The treatment was randomized at the subject level within each session—i.e., within a given experimental session, some subjects were assigned to IDENT and others were assigned to CS. At the end of the experimental session, subjects were randomly and anonymously matched with three other participants in the same treatment. One of the eight games was selected for payment; the same game was selected for the four subjects matched together. Subjects were paid based on their action and the actions of their randomly matched counterparts. Subjects received the GBP value of their payoff in the selected game.

Because the experiment was conducted online, we implemented several features important to preserve anonymity, decrease dropout rates, increase attention, and ensure that subjects understood the experiment.[13] We highlight some of these features. First, we

[12] We ran three lab sessions in March 2020, just before in-person labs closed. The results from our online experiment are in line with the results from the lab sessions. Because the online experiment is a different medium, we do not include the data from the in-person sessions.

[13] *Ex ante*, we could not ensure that there would be no dropout. As such, we had to have an experimental design that was robust to dropout. This is why we matched subjects into groups of four after subjects completed the play of the game. (Note, this does not affect the incentives of participants.) If we end up

adopted a Zoom protocol similar to that used by EBEL at UCSB. (Appendix D describes the protocol.) Second, we had a completion fee of 3.5 GBP. Third, we implemented an incentivized quiz; a screenshot of the quiz can be found in Appendix D. Subjects who answered all the quiz questions correctly on the first try received a bonus of 3 GBP. Moreover, subjects who did not answer the quiz correctly within three tries were automatically assigned to a low-stakes version of the game (played against a computer).[14] Fourth, on each page of the experiment, subjects could both reveal the instructions and anonymously ask questions via a chat box. That is, they did not have to navigate between the experimental interface and the Zoom app to either see instructions or chat with the experimenter.

On average, subjects earned 15 GBP plus the completion fee. In addition 89% of subjects answered all the quiz questions correctly on the first try and, so, also earned the quiz fee. Subjects were paid by bank transfer using Wise.

# 4 Experimental Results

We collected data for 295 subjects. Of those, 147 subjects were assigned to IDENT and 148 subjects were assigned to CS. Thus, there are 295 observations of the form $x = (x(1), x(2), x(3), x(4))$. We use the approach laid out in Section 2 to assign each $x$ a rationality bound (if possible) and a strategic bound (if possible).

Of the 295 subjects, 27 chose a dominated strategy and, thus, we cannot assign a rationality bound to those subjects. Of the remaining 268 subjects, 1 chose strategies that are inconsistent with our assumptions about strategic reasoning and, thus, we cannot assign a strategic bound to that subject. As such, 28 subjects fall outside the purview of our analysis. So, our analysis focuses on the behavior of the remaining 267 subjects.

## 4.1 The Gap

Table 4.1 reports the interaction between the identified rationality and strategic bounds. Observations with an identified gap are those that fall on the off-diagonal. Notice, 159 observations are identified as having a gap. This constitutes 60% of the observations. Table 4.2 highlights that the prevalence of an identified gap is not a treatment-specific effect. In IDENT, 67% of the observations are identified as having a gap, and in CS, 51% of the observations are identified as having a gap. (We will discuss the differences in distributions when we discuss the heuristic beliefs model.)

with a session in which a treatment does not have a multiple of four, we complete the incomplete group with behavior from subjects in a complete group.

[14] Only 1 subject was assigned to the low-stakes game. Because the incentives in that game were quite different from Figures 2.1-2.2, that subject's data is omitted from the dataset.

| | Rationality Bound | | | |
|---|---|---|---|---|
| Strategic Bound | RB=1 | RB=2 | RB=3 | RB=4 |
| SB=1 | 39 | – | – | – |
| SB=2 | 36 | 21 | – | – |
| SB=3 | 17 | 17 | 4 | – |
| SB=4 | 40 | 46 | 3 | 44 |

Table 4.1. Gap Between Identified Rationality and Identified Strategic Bounds

| | Rationality | | | |
|---|---|---|---|---|
| Strategic | RB=1 | RB=2 | RB=3 | RB=4 |
| SB=1 | 15 | – | – | – |
| SB=2 | 20 | 12 | – | – |
| SB=3 | 12 | 8 | 3 | – |
| SB=4 | 27 | 24 | 3 | 16 |

(a) Identification Treatment (IDENT)

| | Rationality | | | |
|---|---|---|---|---|
| Strategic | RB=1 | RB=2 | RB=3 | RB=4 |
| SB=1 | 24 | – | – | – |
| SB=2 | 16 | 9 | – | – |
| SB=3 | 5 | 9 | 1 | – |
| SB=4 | 13 | 22 | 0 | 28 |

(b) Comparative Static Treatment (CS)

Table 4.2. Gap Between Identified Rationality and Identified Strategic Bounds: Treatment

The identified gap reflects a stark contrast between the distribution of identified rationality and strategic bounds. The majority of subjects have a low identified rationality bound: 49% have a rationality bound of 1, and 32% have a rationality bound of 2. By contrast, 50% of subjects have an identified strategic bound of 4.

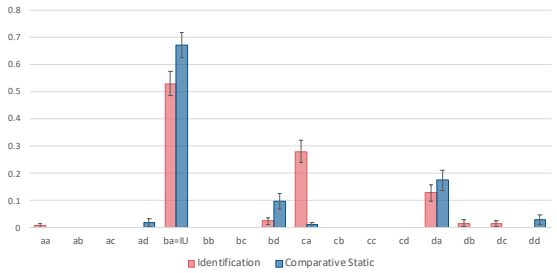
On average, subjects identified as having a gap fare better relative to their no-gap counterparts. To see this, compute a subject's (i.e., observation's) expected P$i$ earnings, given the empirical distribution of observed behavior. (This empirical distribution includes the behavior of all subjects—both classified and non-classified.) Take the average across all player roles P$i$ to compute a subject's average expected earnings. Table 4.3 reports the average earnings across observations identified with a strategic bound of $k$.

Subjects identified with both a strategic bound of $k \geq 3$ and a gap outperform their no-gap counterparts. In fact, for subjects with an identified strategic bound of $k \geq 3$, we can reject the hypothesis that the true average payoffs of the no-gap subjects are greater than or equal to the true average payoffs of their gap counterparts. (A one-sided two-sample $t$-test returns a $p$-value of .0001 for $k = 3$ and .0000 for $k = 4$. We take 'reject' to mean 'reject at the 5% significance level.') However, we cannot reject the hypothesis that, when the identified strategic bound is 2, average payoffs are the same across gap and no-gap subjects. (The two-sided two-sample $t$-test returns a $p$-value=.6123.)

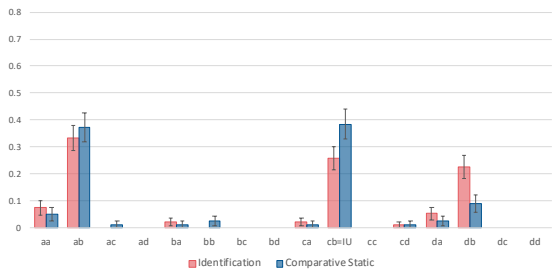
The fact that gap subjects outperform their no-gap counterparts is of interest. Bounds on reasoning about rationality are often interpreted as limits on the ability to engage in interactive reasoning. However, such bounds may instead reflect a deliberate decision to *not* believe that other players are rational (or reason about rationality). The fact that the

| | Identified Strategic Bound | | | |
|---|---|---|---|---|
| | SB=1 | SB=2 | SB=3 | SB=4 |
| No Gap | 14.89 | 15.70 | 14.06 | 13.31 |
| Gap | - | 15.63 | 15.44 | 14.61 |

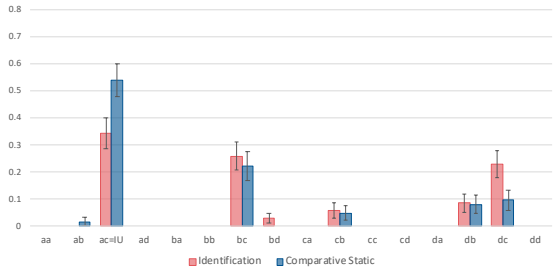
Table 4.3. Average Expected Payoffs




(a) P2 Distribution Given SB $\geq 2$




(b) P3 Distribution Given SB $\geq 3$




(c) P4 Distribution Given SB $\geq 4$

Figure 4.1. P$i$ Distribution of Play Given SB $\geq i$: Error bars represent bootstrapped standard errors

gap subjects outperform their no-gap counterparts may further indicate that some subjects are capable of reasoning about rationality at higher levels, but simply choose not to do so.

## 4.2 Distribution of Play

Refer back to Section 1 (p. 11). There we argued that, if a rational subject is 2-strategic, then, in the role of P2, she may well choose different strategies in IDENT and CS. A similar argument applies to higher levels of strategic reasoning: In player role P$i$=P2,...,P$k$, a rational subject with a strategic bound of $k$ may well choose different strategies in IDENT and CS. So, rational subjects with a strategic bound of $k \geq 2$ *may* induce different distributions of play in the roles of P$i$=P2,...,P$k$.

Figure 4.1 depicts the empirical distribution of P$i$'s play, for subjects with an identified strategic bound $k \geq i$. Note, in each player role P$i$=P2,P3,P4, these empirical distributions are different across treatments. In particular, in each player role, there is substantially

more IU play in CS vs. IDENT. Conversely, in each player role, there is some non-constant strategy $(e, f_*)$ for which there is substantially more $(e, f_*)$ play in IDENT vs. CS. For instance, in the role of P2, 28% of subjects play $(c, a_*)$ in IDENT, but only 1% of subjects play it in CS. Similarly, for $(d, b_*)$ in the role of P3 (23% in IDENT vs. 9% in CS) and $(d, c_*)$ in the role of P4 (23% in IDENT vs. 10% in CS).

While the empirical distributions differ across treatments, they may be different realizations of a single underlying *true* distribution. A chi-square test of homogeneity allows us to test the hypothesis that the two distributions are realizations of the same distribution. We implement a conservative application of this test, one that makes it difficult to reject the hypothesis of homogeneity.[15] We can reject the hypothesis that the true distribution of P2's play (conditional on an identified strategic bound of $k \geq 2$) is independent of the treatment, even if the Bonferroni multiple hypotheses correction is applied. (The $p$-value is .0001.) The same conservative application of the chi-square test does not allow us to reject the hypothesis that the true distribution of P3's (resp. P4's) play conditional on an identified strategic bound of $k \geq 3$ (resp. $k = 4$) is independent of the treatment. (The $p$-values are .6575 and .8405, respectively.) Because we can reject the null for *some* player role, we can conclude that the true distribution of play depends on the treatment.

There is a second implication about the distribution of play. If a rational subject has a strategic bound of $k < 4$, then the subject should play a constant strategy in the role of P$i$=P$(k+1)$,...,P4. Moreover, that constant strategy should not depend on the treatment. So, in player role P$i$, rational subjects with a strategic bound of $k < i$ should induce a distribution of play that is concentrated on constant strategies and independent of the treatment. In fact, this is what we observe. Moreover, these distributions are (almost) degenerate. Subjects with an identified strategic bound of 1 play $(d, d_*)$ in the role of P2, $(a, a_*)$ in the role of P3, and $(b, b_*)$ in the role of P4; this behavior is irrespective of the treatment. Almost all subjects with an identified strategic bound of $k = 2, 3$ play analogously in the role of P$i >$ P$k$.

# 5 Heuristic Beliefs Model

The experimental results highlight that a substantial fraction of subjects have a gap between their identified strategic and identified rationality bounds. As a consequence, there is room for particular forms of strategic reasoning to systematically shape beliefs. This sec-

[15] There are strategies that are not played in either treatment. Zero-frequency strategies cause a problem in computing the chi-square test statistic. The statistic itself be corrected by effectively ignoring zero-frequency strategies. A conservative approach is to, nonetheless, take the number of degrees of freedom to be 1 less than the number of strategies, i.e., 15; this is what we report. A more permissive approach would be to take the number of degrees of freedom to be 1 less than the number of strictly-positive-frequency strategies. Under the permissive approach, the $p$-values are .0000, .1978, and .1396 respectively.

tion explores whether two assumptions about strategic reasoning (namely, simple-strategic beliefs and ordered-heuristic beliefs) do, in fact, shape beliefs.

Sections 5.1-5.2 introduce the theoretical model. Section 5.3 points out that the experimental data is consistent with the model. Sections 5.4-5.6 investigate the comparative statics of the model. Section 5.7 investigates implications for other experimental settings.

### 5.1 Key Assumptions

Strategically optimal behavior can depend on both payoffs and beliefs. Rationality is one such form of strategic optimality. But, there are many forms of strategic optimality that depend on payoffs and not on beliefs. For instance, the lucky-6 rule-of-thumb depends on payoffs but not on beliefs. When a subject's notion of strategic optimality does not depend on her beliefs, we think of the subject as adopting a rule-of-thumb or heuristic.

Simple-strategic beliefs is an assumption that strategic reasoning is shaped by a belief that either other players are rational or they adopt a belief-independent heuristic. That is, P$i$ believes that P$(i-1)$'s strategically optimal but irrational behavior depends *only* on P$(i-1)$'s payoff matrix (and not his beliefs). So, if P$i$ believes that P$(i-1)$ is strategic, she believes that P$(i-1)$ is either rational or P$(i-1)$ plays an invariant strategy.

**Assumption 5.1.** *Pi has a* ***simple-strategic belief*** $\Pr_i$ *if* $\Pr_i(\mathrm{e}, \mathrm{f}_*) > 0$ *implies that either* $(\mathrm{e}, \mathrm{f}_*)$ *is rational or* $(\mathrm{e}, \mathrm{f}_*) \in \mathrm{Inv}_{i-1}$.

Under the assumption of simple-strategic beliefs, a player's belief can be decomposed into a "rational part" and an "invariant (but irrational) part." In the example of Section 1, we implicitly assumed that P2 has a simple-strategic belief. (See footnote 6.)

Ordered-heuristic beliefs pertain to the relative weight that a subject assigns to invariant but irrational strategies. Section 1 explained the idea: If P$(i-1)$ prefers $(f, \Pi_{i-1}(f))$ to $(e, \Pi_{i-1}(e))$ according to a class of natural heuristics, then P$i$ should assign a higher weight $(f, \Pi_{i-1}(f))$ vs. $(e, \Pi_{i-1}(e))$. (Refer back to p. 12.)

Write $(f, \Pi_i(f)) \geq_i^{\max} (e, \Pi_i(e))$ (resp. $(f, \Pi_i(f)) \geq_i^{\min} (e, \Pi_i(e))$) if the maximum (resp. minimum) payoff that P$i$ can achieve by choosing $f$ is at least as high as the maximum (resp. minimum) payoff that P$i$ can achieve by choosing $e$. Likewise, write $(f, \Pi_i(f)) \geq_i^{\mathrm{sum}} (e, \Pi_i(e))$ if the sum of P$i$'s payoffs associated with f is at least as high as the sum of P$i$'s payoffs associated with e. Say $(f, \Pi_i(f))$ **heuristic dominates** $(e, \Pi_i(e))$ for P$i$ if $(f, \Pi_i(f)) \geq_i^{\max} (e, \Pi_i(e))$, $(f, \Pi_i(f)) \geq_i^{\min} (e, \Pi_i(e))$, and $(f, \Pi_i(f)) \geq_i^{\mathrm{sum}} (e, \Pi_i(e))$; in that case, write $(f, \Pi_i(f)) \geq_i^{\square} (e, \Pi_i(e))$. Note, the games were designed so that, for each P$i$=P1,P2,P3, $\geq_i^{\max}, \geq_i^{\min}, \geq_i^{\mathrm{sum}}$ are complete orders that coincide. So, for P$i$=P1,P2,P3, $\geq_i^{\square}$ is a complete order.[16]

[16] More precisely, $\geq_i^{\max}$ and $\geq_i^{\mathrm{sum}}$ are complete and strict orders that coincide. Moreover, for P$i$=P1,P2,P3,

**Assumption 5.2.** *Pi has an* ***ordered-heuristic beliefs*** *if Pi has a simple-strategic belief* $\Pr_i$ *and the following holds: If* $(f, \Pi_{i-1}(f))$ *heuristic-dominates* $(e, \Pi_{i-1}(e))$ *and* $(e, \Pi_{i-1}(e))$ *is irrational, then* $\Pr_i(f, \Pi_{i-1}(f)) \geq \Pr_i(e, \Pi_{i-1}(e))$.

The assumption of heuristic beliefs requires that, if P$i$ assigns positive probability to an irrational strategy $(e, \Pi_{i-1}(e))$ and $(f, \Pi_{i-1}(f))$ has a higher maximum/minimum/sum than $(e, \Pi_{i-1}(e))$, then P$i$ assigns higher probability to $(f, \Pi_{i-1}(f))$ over $(e, \Pi_{i-1}(e))$. We refer to a belief that satisfies this condition as an *ordered-heuristic belief*. (Note, an ordered-heuristic belief is, by definition, a simple-strategic belief.)

Let us pick up on two comments made in Section 1 about the ordered-heuristic beliefs assumption. First, the assumption rests on three rules of thumb: maximum, minimum, and sum. We think that each of these rules of thumb is reasonable. For P$i$=P1,P2,P3, the relations $\geq_i^{\max}$, $\geq_i^{\min}$, $\geq_i^{\text{sum}}$ coincide; as such, we need not "choose" between the criteria. (Again, this feature is by design.) Instead, we seek to understand whether or not they jointly impact strategic reasoning. Second, we also don't rule out the possibility that other rules of thumb may be at play. In fact, that very possibility is why we allow $\Pr_i(e, \Pi_{i-1}(e)) > 0$ even if $(f, \Pi_{i-1}(f))$ heuristic dominates $(e, \Pi_{i-1}(e))$. Instead, we view these rules of thumb as being prominent in how players engage in strategic reasoning. Third, we do not insist that these rules of thumb are, in fact, an important determinant of beliefs. Instead, we seek to allow the data to weigh in on this question.

## 5.2 Heuristic Reasoning and Bounds

Say a subject is *1-heuristic* if, in each player role, she is strategic. Say a subject is *2-heuristic* if she is 1-heuristic and, in each player role, (i) she has an ordered-heuristic belief, and (ii) she believes that, if a subject is rational, the subject is not indifferent between any two strategies. For $k \geq 3$, a subject is *$k$-heuristic* if she is $(k-1)$-heuristic and believes that others are $(k-1)$-heuristic. She has a *heuristic bound of 1* if she is 1-heuristic and satisfies the Principle of Non-Strategic Reasoning. Inductively, a subject has a *heuristic bound of $k$* if she is $k$-heuristic and believes that others have a heuristic bound of $(k-1)$.

If a subject is $k$-heuristic, the subject is also $k$-strategic; likewise, if a subject has a heuristic bound of $k$, the subject also has a strategic bound of $k$. (See Lemmata B.1-B.2.) The key is that, if a subject has a simple-strategic belief, then she must believe that others satisfy the Principle of Strategic Reasoning. So, this new model of strategic reasoning—the

$(f, \Pi_i(f)) \geq_i^{\max} (e, \Pi_i(e))$ implies $(f, \Pi_i(f)) \geq_i^{\min} (e, \Pi_i(e))$. So, for P$i$=P1,P2,P3, $\geq_i^{\square}$ is a complete strict order that coincides with $\geq_i^{\max}$=$\geq_i^{\text{sum}}$. Note, $(f, \Pi_4(f)) \geq_4^{\max} (e, \Pi_4(e))$ does not imply $(f, \Pi_4(f)) \geq_4^{\min} (e, \Pi_4(e))$; as such, $\geq_4^{\square}$ is not a complete order. However, because only P1 cares about P4's behavior and we restrict attention to rational observations, this will not matter.

**heuristic beliefs** (**HB**) **model**—is nested in the original model of strategic reasoning—what we now refer to as the **strategic beliefs** (**SB**) **model**.

Inductively define $i$-heuristic beliefs, an analogue to 2-strategic beliefs. Set $\text{HB}_1 = \{(\text{a}, \text{d})\}$, i.e., the set that contains the dominant strategy for P1. Call $\Pr_i$ a *$i$-heuristic belief* if it is an ordered-heuristic belief of P$i$ that assigns probability 1 to $\text{HB}_{i-1} \cup \text{Inv}_{i-1}$. Set $\text{HB}_i$ as the set of strategies of P$i$ that are a unique best response under an $i$-heuristic belief.

**Identification.** *Identify $(x(1), x(2), x(3), x(4))$ as having a heuristic bound of 1 if it is identified it as having a strategic bound of 1. Identify $(x(1), x(2), x(3), x(4))$ as having a heuristic bound of $k \geq 2$ if it has not been identified as having a heuristic bound of $j < k$ and the following hold:*

*(i) for each $j = 1, \ldots, k$, $x(j) \in \text{HB}_j$; and*

*(ii) for each $j > k$, $x(j)$ is a constant strategy.*

An observation is identified as having a heuristic bound of $k \geq 2$ if and only if the observation is consistent with a heuristic bound of $k$ and there is no other $j < k$ so that the observation is consistent with a heuristic bound of $j$. Proposition B.1 shows that this identification corresponds to the minimal heuristic bound consistent with observed behavior, provided a $k$-heuristic subject satisfies $(k-1)^{\text{th}}$-order belief of non-indifference.

### 5.3 Experimental Results

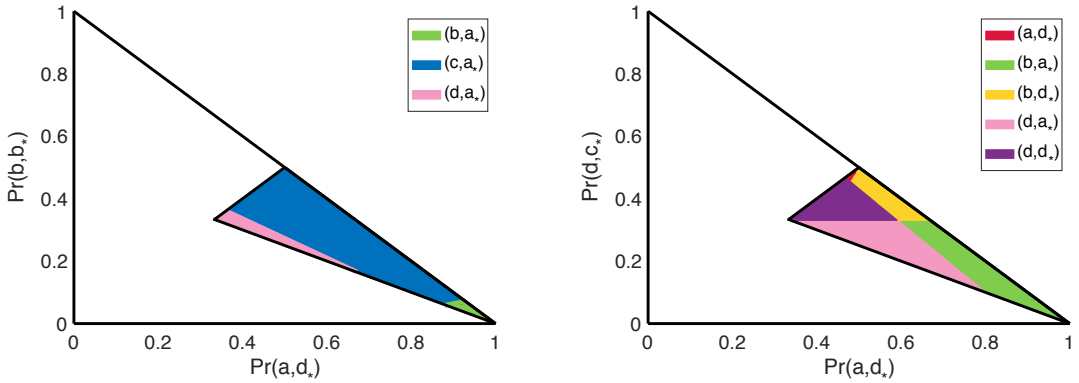
If an observation is identified as having a heuristic bound of $k$, it is identified as having a strategic bound of $k$. (See Lemma B.3.) But, an observation can be identified as having a strategic bound of $k$ even if it is not identified as having any heuristic bound. So, the HB model provides tighter predictions than the SB model. (See Tables C.2-C.6.)

In fact, fewer than 1% of the potential observations are consistent with the HB model. That is, many observations would not be identified with a heuristic bound. Despite the tight set of predictions, 88% of rational observations are consistent with the HB model.

Moreover, the HB model generates predictions for cross-treatment differences in the distribution of play and cross-treatment differences in the identified rationality bounds. We next discuss those predictions and how they align with our experimental results.

### 5.4 Distribution of Play

If a rational subject has a heuristic bound of $k \geq i \geq 2$, then the subject's behavior in the role of P$i$ lies in $\text{HB}_i$. Since a 2-heuristic belief depends on the treatment, the set $\text{HB}_2$ varies with the treatment. As a consequence, for each $k \geq 3$, the sets $\text{HB}_k$ vary by treatment.




(a) Identification (IDENT): $\Pr(\mathrm{d}, \mathrm{c}_*) = 0$ (b) Comparative Static (CS): $\Pr(\mathrm{b}, \mathrm{a}_*) = 0$

Figure 5.1. P2's Beliefs and Best Responses

**Distribution of P2's Play** A strategy of P2 is in $\mathrm{HB}_2$ if it is a unique best response to a 2-heuristic belief. There are two properties that each imply that the set of 2-heuristic beliefs varies by treatment. First, because a 2-heuristic belief is a simple-strategic belief, it assigns probability $p \in [0, 1]$ to the dominant strategy $(\mathrm{a}, \Pi_1(\mathrm{a}))$ and probability $(1 - p)$ to the irrational but invariant $\{(\mathrm{b}, \Pi_1(\mathrm{b})), (\mathrm{c}, \Pi_1(\mathrm{c})), (\mathrm{d}, \Pi_1(\mathrm{d}))\}$. This irrational invariant set differs across treatments since $\Pi_1$ differs across treatments. Second, because a 2-heuristic belief is an ordered-heuristic belief, it assigns weight in accordance with P1's heuristic dominance order. The heuristic dominance order differs across treatments—above and beyond the permutation changing—since the rows b and d are flipped for P1.

To make this more concrete, suppose P2's belief assigns probability 0 to the "heuristic worst" strategy—i.e., the strategy that is "worst" according to P1's heuristic dominance order. So, in IDENT, $\Pr_2(\mathrm{d}, \Pi_1(\mathrm{d})) = 0$, and in CS, $\Pr_2(\mathrm{b}, \Pi_1(\mathrm{b})) = 0$. Under this assumption, we can draw P2's belief in a two-dimensional simplex. Figure 5.1 depicts this simplex; any point in the large triangle specifies the probability that P2 assigns to each of the remaining invariant strategies. For both treatments, the $x$-axis represents the probability that P2 assigns to the rational $(\mathrm{a}, \mathrm{d}_*)$. This is also the probability that P2 assigns to the "heuristic best" strategy. For both treatments, the $y$-axis represents the probability that P2 assigns to the "heuristic second-best" strategy and the distance from the diagonal represents the probability that P2 assigns to the "heuristic third-best" strategy. So, the vertex $(1, 0)$ corresponds to P2 assigning probability 1 to the IU strategy; the vertex $(0, 1)$ corresponds to P2 assigning probability 1 to the "heuristic second-best;" and the vertex $(0, 0)$ corresponds to P2 assigning probability 1 to the "heuristic third-best." Importantly, the treatments differ in what is the "heuristic second-best" and "heuristic third-best" strategies.

The ordered-heuristic beliefs assumption adds the requirement that the probability as-

signed to the $x$-axis must be weakly higher than the probability assigned to the $y$-axis which, in turn, must be weakly higher than the probability assigned to the diagonal. So, the set of beliefs consistent with the ordered-heuristic beliefs assumption corresponds to the small triangles in Figure 5.1 (outlined in black). While these are each ordered-heuristic beliefs, they assign different probabilities to P1's strategies since P1's matrix has changed. This has implications for P2's best response: In each treatment, P2 may have an ordered-heuristic belief that assigns probability $p$ (resp. $q$) to the heuristic-best strategy (resp. heuristic-second-best strategy) and, despite this, P2 may have a different best response across the treatments. This can be seen from the colored areas, which represent P2's best responses. In particular, there are points in the triangle at which the best response differs across treatments. Moreover, the set of all best responses also differs across treatments: In IDENT, the strategies that are a unique best response under some ordered-heuristic belief are $(\mathrm{b}, \mathrm{a}_*)$, $(\mathrm{c}, \mathrm{a}_*)$, and $(\mathrm{d}, \mathrm{a}_*)$; in a sense, $(\mathrm{c}, \mathrm{a}_*)$ is the most prominent among these. In CS, the strategies that are a unique best response are $(\mathrm{a}, \mathrm{d}_*)$, $(\mathrm{b}, \mathrm{a}_*)$, $(\mathrm{b}, \mathrm{d}_*)$, $(\mathrm{d}, \mathrm{a}_*)$, and $(\mathrm{d}, \mathrm{d}_*)$.

The message remains analogous, even if P2 holds an ordered-heuristic belief that assigns positive probability to the "heuristic worst" outcome. Figure D.1 varies the probability assigned to the "heuristic worst" outcome. In each treatment, the set of non-constant strategies that are a best response under some ordered-heuristic belief is the same as in Figure 5.1. (The only addition is that the constant $(\mathrm{d}, \mathrm{d}_*)$ can be a best response to an ordered-heuristic belief in IDENT.) So, put together, these figures depict $\mathrm{HB}_2$.

The predictions accord with the observed distributions of P2 play, conditional on being identified as having a strategic bound of $k \geq 2$. Refer to Figure 4.1a. First, we observe that more subjects play $(\mathrm{c}, \mathrm{a}_*)$ in IDENT (28%) vs. CS (1%); the strategy $(\mathrm{c}, \mathrm{a}_*)$ is a best response (under a 2-heuristic belief) only in IDENT. Second, we observe that more subjects play the IU strategy, $(\mathrm{b}, \mathrm{a}_*)$, in CS (67%) vs. IDENT (53%); the IU strategy is a "more prominent" best response in CS. More generally, strategies that are played often are in $\mathrm{HB}_2$. But, more notably, the strategies that are not in $\mathrm{HB}_2$ are rarely played—this is the case despite the fact that about $3/4$ of the feasible strategies are not in $\mathrm{HB}_2$. In IDENT (resp. CS), only 6% (resp. 1%) of the strategies played are not in $\mathrm{HB}_2$.

**Distribution of P3's and P4's Play** Now turn to behavior in the role of P$i$, where $i \geq 3$. To do so, consider a rational subject with a heuristic bound of $k \geq i$. Since P$i$ has a simple-strategic belief, any strategy that gets strictly positive probability must be a strategy that is either rational for P$(i-1)$ or invariant for P$(i-1)$. Since the subject has a heuristic bound of $i$, P$i$ also believes that the rational strategies of P$(i-1)$ are contained in $\mathrm{HB}_{i-1}$. As such, in the role of P$i$, the subject must assign probability 1 to

$$\mathrm{HB}_{i-1} \cup \mathrm{Inv}_{i-1} = \mathrm{HB}_{i-1} \cup \{(\mathrm{a}, \mathrm{a}_*), (\mathrm{b}, \mathrm{b}_*), (\mathrm{c}, \mathrm{c}_*), (\mathrm{d}, \mathrm{d}_*)\}.$$

Now notice that, because $HB_2$ varies by treatment, the beliefs of such a P3 will vary by treatment. As such $HB_3$ varies by treatment. An analogous argument implies that $HB_4$ varies by treatment. The sets $HB_3$ and $HB_4$ can be read from Tables C.5-C.6.

The predictions accord with the observed distributions of play. To see this, focus on the behavior in the role of P3 (resp. P4) for subjects with an identified strategic bound of $k \geq 3$ (resp. $k = 4$). There are three notable features. First, in the role of P3 (resp. P4), we observe more subjects play $(\mathrm{d}, \mathrm{b}_*)$ (resp. $(\mathrm{d}, \mathrm{c}_*)$) in IDENT vs. in CS. This is intuitive under the HB model. For P3, the strategy $(\mathrm{d}, \mathrm{b}_*)$ is in $HB_3$ for IDENT but not CS. Moreover, in IDENT, there is exactly one degenerate belief under which the strategy $(\mathrm{d}, \mathrm{b}_*)$ is a best response—a belief that assigns probability 1 to $(\mathrm{c}, \mathrm{a}_*)$. As pointed out above, $(\mathrm{c}, \mathrm{a}_*)$ is a best response under a 2-heuristic belief in IDENT—indeed, one often played in IDENT. Likewise, for P4, there is exactly one degenerate belief under which the strategy $(\mathrm{d}, \mathrm{c}_*)$ is a best response—a belief that assigns probability 1 to $(\mathrm{d}, \mathrm{b}_*)$. This is a 4-heuristic belief in IDENT but not in CS.[17] Second, we observe more subjects play the IU strategy in CS; as we will explain in the next subsection, this is natural given that, for P2, IU is a "more prominent" best response in CS. Third, strategies not in $HB_3$ or $HB_4$ are not played often. For P3 (resp. P4), only 11% (resp. 5%) of the strategies played are not in $HB_3$ (resp. $HB_4$).

In sum, because $HB_2$ varies by treatment, we would expect to see differences in the distribution of play across treatments. The experimental results point to such differences. Moreover, the observed differences are consistent with the HB model (and the SB model).

### 5.5 Identified Rationality Bounds

Assume that subjects' *actual* rationality bounds do not vary across treatments. Under the HB model, we would expect IDENT to be associated with lower *identified* rationality bounds relative to CS. Section 1 (p. 13) pointed to the basic idea. We now expand.

Consider a rational subject whose heuristic bound is $k \geq 2$. First focus on behavior in the role of P2. Because P2 has an ordered-heuristic belief, the belief can be seen as one that assigns probabilities $p : q : r$ to the heuristic best, second-best, and third-best strategies. So, P2 assigns probability $p$ to P1's dominant strategy. The IU strategy is a best response to such a belief, if $p$ is "sufficiently high."

Suppose P2 forms a 2-heuristic belief systematically across treatments—that is, P2 assigns the same probabilities $p : q : r$ to the heuristic best, second-best, and third-best strategies. (Again, those strategies change with the treatment.) In that case, the IU strategy is a best response in IDENT, only if it is also a best response in CS. (See Figures 5.1 and

[17] The strategy $(\mathrm{d}, \mathrm{c}_*)$ is a best response under a belief that is a 4-heuristic belief in CS; however, any such belief must be non-degenerate assigning positive probability to both $HB_3$ and $Inv_3$.

| | Rationality Bound | | | |
|---|---|---|---|---|
| Treatment | RB=1 | RB=2 | RB=3 | RB=4 |
| Identification (IDENT) | 53% | 31% | 4% | 11% |
| Comparative Static (CS) | 46% | 31% | 1% | 22% |

Table 5.1. Rationality bounds ($p = .0391$)

D.1.) Thus, if P2's 2-heuristic belief is determined systematically across treatments, P2 would play the IU strategy in IDENT only if P2 would also play the IU strategy in CS. (If P2 has nearby beliefs, a similar conclusion would hold.) But the converse does not hold: For the IU strategy to be a best response in IDENT, P2 must assign $p \geq \frac{121}{158}$ to the heuristic-best strategy. However, in CS, the IU strategy *can* be a best response if P2 only assigns probability $p \geq \frac{80}{137}$ to the heuristic-best strategy.

The analysis of P3 yields analogous cross-treatment differences. The key is that a 3- or 4-heuristic P3 equates "P2 is rational" with "P2 plays a best response to a 2-heuristic belief." If such a P3 thinks that P2 forms his 2-heuristic belief systematically across treatments, the probability that P3 assigns to the event "P2 plays the IU strategy" will be higher in CS vs. IDENT. Thus, P3 is more prone to play IU in CS vs. IDENT. And similarly for P4.

Indeed, this is what we observe in the data. Table 5.1 implies that the empirical CDF of identified rationality bounds in CS first-order stochastically dominates the empirical CDF of identified rationality bounds in IDENT. IDENT is associated with a larger (resp. smaller) fraction of subjects with an identified rationality bound of 1 (resp. 4). A chi-square test of homogeneity allows us to reject the hypothesis that the *true* distributions of (identified) rationality bounds are the same across treatments. (The associated $p$-value is .0391.) Moreover, a one-sided two-sample t-test allows us to reject the hypothesis that the true mean of identified rationality bounds in IDENT is at least as high as the true mean of identified rationality bounds in CS. (The associated $p$-value is .0481.)

## 5.6 Identified Gap

While the identified rationality bounds vary by treatment, we cannot rule out the possibility that the true distributions of identified strategic bounds are the same across treatments. Refer to Table 5.2a, which depicts the empirical distribution of identified strategic bounds. (A chi-square test of homogeneity is associated with a $p$-value of .2245; so we cannot reject the hypothesis of homogeneity.) Referring to Table 5.2b, a similar message pertains to the identified heuristic bounds.

This suggests that, if subjects hold heuristic beliefs, we would expect to see a larger identified gap in IDENT relative to CS. Indeed, that was the message of Table 4.2. In IDENT, 67% of observations were identified as having a gap, and in CS, 51% of observations

| Strategic Bound | | | | |
|---|---|---|---|---|
| Treatment | SB=1 | SB=2 | SB=3 | SB=4 |
| Identification (IDENT) | 11% | 23% | 16% | 50% |
| Comparative Static (CS) | 19% | 20% | 12% | 50% |

(a) Strategic Bounds ($p = .2245$)

| Heuristic Bound | | | | |
|---|---|---|---|---|
| Treatment | HB=1 | HB=2 | HB=3 | HB=4 |
| Identification (IDENT) | 12% | 23% | 15% | 50% |
| Comparative Static (CS) | 21% | 22% | 11% | 47% |

(b) Heuristic Bounds ($p = 0.2558$)

Table 5.2. Empirical Distributions of Bounds by Treatment

were identified as having a gap. These results further suggest that the identified gap in CS may be an underestimate of the actual gap.

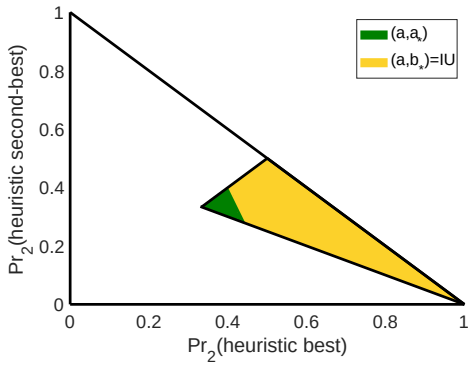
## 5.7 Other Ring Game Data

The data in Kneeland (2015) and Sprenger and Zhao (2021) serve to provide out-of-sample predictions. This section looks at the implications of the HB model for their data. Both papers feature permuted ring games in which players choose one of three actions. In our discussion, we order the players so that, as in this paper, P1 has a dominant strategy; P2's payoffs depend only on P1's and P2's behavior; etc.

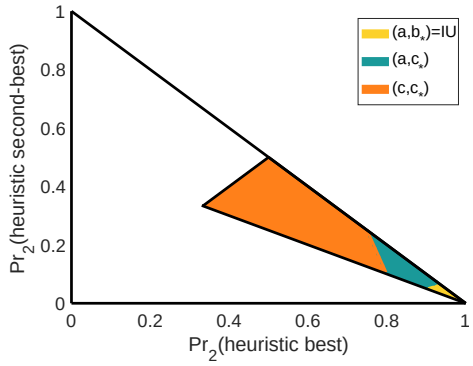
Note, the full dataset from Sprenger and Zhao is not publicly available. As such, we will not be able to identify strategic or rationality bounds for any given observation. Yet, their results allow us to infer observed behavior within a given player role. So, we will limit statements about Sprenger and Zhao to statements about this behavior.

**Distribution of Play** In Kneeland's data, 81% of rational subjects have behavior consistent with HB. Sprenger and Zhao's data also appears consistent with HB; at least 90% of subjects choose (within player role) strategies that are consistent with HB.

**Identified Rationality Bounds** Under the HB model, we would expect Sprenger and Zhao's data to be associated with lower rationality bounds relative to Kneeland's data. To see this, consider a rational subject whose heuristic bound is $k \geq 2$. Suppose she forms her 2-heuristic belief systematically across treatments. So, in the role of P2, she always assigns the same probability $p : q$ to the heuristic best : second-best strategies. Refer to Figure 5.2a and notice that the IU strategy is a best response in Sprenger and Zhao only if it is




(a) Kneeland's (2015) Game




(b) Sprenger and Zhao's (2021) Modified Game

Figure 5.2. P2's Beliefs and Best Responses

| | |
|---|---|
| Kneeland (2015) | 76% |
| Comparative Static Treatment | 57% |
| Identification Treatment | 48% |
| Sprenger and Zhao (2021) | $< 18\%$ |

Table 5.3. Percentage of Rational Subjects that Play IU in P2

a best response in Kneeland. Thus, the subject plays the IU strategy in the role of P2 in Sprenger and Zhao only if she does the same in Kneeland.

We get a more complete picture by comparing the IU play in Kneeland and Sprenger and Zhao to that in IDENT and CS. Under the HB model, we would expect CS to be associated with lower rationality bounds relative to Kneeland's data and we would expect Sprenger and Zhao's to be associated with lower rationality bounds relative to IDENT. To understand why, fix a treatment (or experiment) $T$ and let $\underline{p}^T$ be the infimum of the set of $p \in [0,1]$ so that the following holds: There is a 2-heuristic belief $\Pr_2$ where (i) P2 assigns probability $p$ to P1's IU strategy, and (ii) P2's strict best response under $\Pr_2$ is his IU strategy. Thus, $\underline{p}^T$ is the minimum probability that P2 can assign to the heuristic best (i.e., rational) strategy of P1 and still have the IU strategy as a best response. The key is that

$$\underline{p}^{SZ} > \underline{p}^{\text{IDENT}} > \underline{p}^{\text{CS}} > \underline{p}^{K},$$

where SZ denotes the Sprenger and Zhao experiment and K denotes the Kneeland experiment. (To see this, refer to Figures 5.1, D.1, and 5.2.) This suggests that, if subjects systematically determine the probability that they assign to the heuristic best (or rational) strategy, then we should see more IU behavior in Kneeland than in CS and more IU behavior in IDENT than in Sprenger and Zhao.

Table 5.3 compares the rates of IU play in the role of P2. It is higher in Kneeland than in CS and it is higher in IDENT than in Sprenger and Zhao, as predicted.

The above logic would suggests a cross player-role prediction: The CDF of identified rationality bounds in Kneeland should first-order stochastically dominate the CDF of identified rationality bounds in CS. The empirical CDFs marginally fail this criterion: The proportion of subjects with an identified rationality bound of 4 is slightly higher in Kneeland vs. CS. Slightly higher is less than 1% higher. So, it is feasible that the true CDFs might satisfy this criterion. (Ideally, we would have a test that would say we could not rule out the hypothesis of first-order stochastic dominance. But we do not know of such a test.)

**Identified Gap** If subjects' actual strategic bounds do not vary across datasets, then the predictions of the identified rationality bounds have implications for the identified gap. In particular, we would expect to see a lower identified gap in Kneeland than in CS and a higher identified gap in Sprenger and Zhao than in IDENT. Because we do not have access to the individual observations in Sprenger and Zhao, we can only test the hypothesis in Kneeland. We observe a smaller identified gap in Kneeland relative to CS. Of the classified subjects, 31% are identified as having a gap between their strategic and rationality bounds.

# 6 Data Generating Process: Alternate Models

The identification strategy rests on an implicit assumption that observed behavior is a consequence of a deliberate (or rational) choice by subjects that engage in strategic reasoning. Could an alternate model of decision making generate the observed data? This section addresses why the data could not be generated by three prominent models.

## 6.1 Errors in Decision Making

One hypothesis is that the observed data is driven by errors in decision making. Subjects may not pay attention; they can misunderstand the game or interface; they can mistakingly click on the wrong choice in the computer interface; they may experience fatigue; etc. Such subjects may then make "incorrect" or "irrational" choices.

These sources of errors are certainly prevalent in decision making. However, the extent to which they influence observed behavior, presumably, depends on the experimental design. Our design took a number of steps to mitigate these issues. To highlight some important design features: First, we implemented a quiz that had both high- and low-stakes incentives. Second, each screen allowed subjects to access the instructions without leaving the screen/game. Third, each screen allowed subjects to anonymously ask questions through a chat box. Fourth, subjects had to spend at least 90 seconds on each decision screen. Fifth, after making their initial choices, subjects were given the opportunity to revisit their choices and make revisions; importantly, the opportunity to revise was not known *ex ante*.

Sixth, subjects only had eight decisions to make, mitigating the potential for fatigue. (See Appendix D for details of the design.)

Despite these measures, it is reasonable to expect some errors in decision making. However, these errors alone could not generate the observed behavior. We now explain why.

**Comparative Static** If the observed distribution of play were largely determined by errors in decision making, we would expect to see the same distribution of play in both IDENT and CS. To understand why, first, consider errors that arise from misunderstanding the game, making the wrong choice in the computer interface, fatigue, etc. There is no reason to hypothesize that such mistakes would depend on the treatment (or even the game). Second, suppose there are mistakes in decision-making that depend on a subject's own payoffs, as in the (prominent) logistic choice model. Because P2, P3, and P4's payoff functions are game-independent and the $m$-undominated strategies are independent of the treatment, the nature of these mistakes should not depend on the treatment. So, in either case, we would expect to see the same distribution of play across IDENT and CS.

But, the empirical distributions of play in IDENT and CS are different. Moreover, we saw that the observed differences in empirical distributions cannot arise from a single true distribution of play. The implication is that differences in distributions cannot solely be an artifact of errors—the two distributions must be fundamentally different.

**Observed Non-Constant Play as Errors** Figure 4.1 highlights that there are many non-constant strategy profiles not observed in the data. If the data were generated by errors, presumably we would see other non-constant play.

Appendix E formalizes this idea. It begins with the premise that there is no gap between the players' rationality and strategic bounds; so, all non-constant play must be an artifact of errors. With this in mind, it specifies a model of $m$-types that play $m$ rounds of IU subject to errors. (Formally, it looks at two such models, based on two different distributions of errors: random choice and logistic errors.) It uses the data to estimate the best fitting error model and uses that model to simulate the expected distribution of play. Doing so indicates that we should expect much more (i) non-constant behavior and (ii) behavior inconsistent with both the SB and HB models. Thus, the data cannot be explained by errors.

**Estimating the Gap** The previous comment leaves open the possibility that the data is partially explained by the SB or HB models and partially explained by errors. This would allow for the possibility that, by analyzing the data without accounting for errors, we may be overestimating the gap. To address this, we estimate an HB model with errors. Appendix E provides details of the model and the estimation. We focus on a model parameterized by

$m$-types and $k$-HB types: $m$-types play $m$ rounds of IU subject to errors; $k$-HB types play in accordance with a heuristic bound of $k$ modulo errors. We then use the data to estimate the best fitting model. In that model, around 60% of subjects are classified as having a gap between their rationality and strategic bounds. This fits with the gap identified in Section 4.1. As such, it suggests that errors do not lead us to overestimate the gap.

**Errors Across Player Roles** Errors that arise from misunderstanding the game, making the wrong choice in the computer interface, fatigue, etc., would, presumably, be independent of the player role.[18] So, presumably, subjects are equally likely to make errors in each player role. Moreover, there is an argument to be made that measurement error is easier to detect in the role of P1 (vs. P2, P3, or P4). So, observed behavior in the role of P1 can provide an upper bound on the magnitude of errors across player roles. Note, only 9% of subjects play a dominated strategy in the role of P1. Given that 60% of classified subjects have an identified gap, it seems difficult to imagine that the errors account for the identified gap.

### 6.2 Random Utility

Another hypothesis is that the observed non-constant non-IU behavior arises from players who have random utility preferences. In principle, such random utility (RU) can lead to stochastic choice. However, RU alone could not generate the behavior observed here. We now explain why.

Begin by reviewing the standard setup: expected utility maximization. There, the subject has a monotonic Bernoulli utility function over monetary outcomes. In each decision round, the subject forms a subjective belief about the behavior of other subjects and chooses an action that maximizes her subjective expected payoffs.

There is a natural analogue for a RU subject. The subject now has a set of permissible Bernoulli utility functions, $\mathcal{U}$, where each $u \in \mathcal{U}$ is strictly increasing in monetary prizes.[19] Note, the distribution of $\mathcal{U}$ is independent of the game. In the spirit of Agranov and Ortoleva (2017), suppose that, in each decision round, the subject draws a new Bernoulli utility function from $\mathcal{U}$, forms a belief about the behavior of other subjects, and chooses an action.[20] Because we seek to understand whether RU *alone* could generate the distribution of observed behavior, we only allow subjects to either believe "rationality" or to believe "other subjects are non-strategic."

[18] Note, the order of play was random. Thus, the player role should be independent of factors like fatigue or misunderstanding the game.

[19] This is a standard substantive assumption—one also adopted in Agranov and Ortoleva (2017).

[20] Note, if the subject only draws a utility function at the start of the experiment, then the analysis is identical to the subjective expected utility benchmark.

Focus on a subject in the role of P2. First, suppose P2 believes P1 is "rational." This allows for the possibility that P2 believes P1 has random utility and maximizes her subjective expected payoff given the realization of utility. But, even so, P2 must believe that P1 chooses her dominant strategy. (This is because the permissible Bernoulli utility functions are monotonic.) As such, even if P2 has random utility, P2 would also choose the IU strategy. Second, suppose P2 believes P1 is "not strategic." Now, P2 has a constant belief. But, because P2 has different utility functions in $G$ and $G_*$, he may well choose a non-constant strategy. That said, there are observable implications of P2's behavior.

**Non-Constant Strategies** In this setting, non-constant non-IU behavior can only be generated by RU subjects that believe others are not strategic. If these were the only RU subjects, we would expect to see the same distribution of play across $G$ and $G_*$. But, because there may be RU subjects that believe "rationality," there may be RU subjects that play the IU $(\mathrm{b}, \mathrm{a}_*)$. This can lead the fraction of subjects that play b in $G$ to be different from the fraction of subjects that play $\mathrm{b}_*$ in $G_*$. (And, similarly, with a and $\mathrm{a}_*$.) However, we would expect the same fraction of subjects to play c in $G$ and $\mathrm{c}_*$ in $G_*$. (And, similarly, with d and $\mathrm{d}_*$.)

With this in mind, we focus on actions e where, in the role of P$i$, neither $(\mathrm{e}, \mathrm{f}_*)$ nor $(\mathrm{f}, \mathrm{e}_*)$ are IU. (Note, within any treatment, there are six such actions, i.e., two for each P$i$=P2,P3,P4.) The RU model generates the hypothesis that the fraction of subjects that play e in the role of P$i$ is the same as the fraction of subjects that play $\mathrm{e}_*$ in the role of P$i$. A two-sided paired $t$-test allows us to reject this hypothesis. (The test returns the following $p$-values: In the role of P2, .0000 for c and .0058 for d; in the role of P3, .0000 for a and .0000 for d; in the role of P4, .0199 for b and .0000 for d.) Moreover, we can reject five of these six hypotheses, even if the Bonferroni multiple hypotheses correction is applied. (The only hypothesis that cannot be rejected after the correction is that, in the role of P4, the fraction of subjects that play b is the same as the fraction of subjects that play $\mathrm{b}_*$.) Because we can reject five of the six hypotheses, we can conclude that the non-constant non-IU play is not due to RU alone.

**Comparative Static** Return to P2. Note, if all subjects were RU and believe either "rationality" or "P1 is not strategic," we would expect to see the same distribution of play across treatments. This is because the IU strategy remains unchanged across treatments and the distribution of $\mathcal{U}$ is independent of the treatment. An analogous argument applies to other player roles. Yet, we have seen that there are systematic differences in behavior across treatments.

**Revisions** The experimental design allowed subjects to revisit their earlier decisions and revise them, should they desire to do so. (This was, *ex ante*, unknown to the subjects.) Much as in Agranov and Ortoleva (2017) and Dwenger, Kübler, and Weizsäcker (2018), if behavior were driven by different draws of utility functions, we would expect subjects to draw new distributions at the revision stage and, so, revise their behavior. Yet, only 14% of choices involve revision. (There are 295 subjects who each make 8 choices; thus, there are 2360 choices made. Of those, 14% get revised.)

### 6.3 Level-$k$ and Cognitive Hierarchy

A final hypothesis is that the observed behavior is generated by either of the level-$k$ (Nagel, 1995; Stahl and Wilson, 1994; Costa-Gomes, Crawford, and Broseta, 2001; Costa-Gomes and Crawford, 2006) or cognitive hierarchy (Camerer, Ho, and Chong, 2004) models. However, these models cannot generate the behavior observed here. We now explain why.

Both the level-$k$ and cognitive hierarchy models begin with an exogenous distribution on strategies played, i.e., the distribution of level-0 behavior. This level-0 behavior is meant to capture the behavior of subjects (real or fictional) that are "non-strategic." As such, it makes sense to assume that the distribution is independent of the game and the treatment. (This is in the spirit of how the model is typically applied.) Given this assumption, both the level-$k$ and cognitive hierarchy models preclude certain observed behavior.

**Distribution of Play** First focus on the level-$k$ model. A level-1 player chooses the IU strategy in the role of P1 and a particular constant strategy in the roles of P2,P3,P4. More generally, given $k \in \{1, 2, 3, 4\}$, a level-$k$ player will choose (a) the IU strategy in the role of $\mathrm{P}1, \ldots, \mathrm{P}k$, and (b) if $4 > k$, a particular constant strategy in the role of $\mathrm{P}(k+1), \ldots, \mathrm{P}4$. This precludes the non-constant non-IU behavior, prominent in the experimental results.

In the cognitive hierarchy model, P2 chooses a best response to a belief that assigns probability $p$ to P1 playing the level-0 distribution and probability $(1-p)$ to P1 playing IU. When level-0 is the uniform distribution (as is customary), certain observed non-constant strategies are precluded, e.g., in the role of P2, $(\mathrm{c}, \mathrm{a}_*)$. This is notable because, in IDENT, 28% of subjects with an identified strategic bound of $k \geq 2$ play $(\mathrm{c}, \mathrm{a}_*)$ in the role of P2.

**Comparative Static** Because the distribution of level-0 behavior is the same in both $G$ and $G_*$, both the level-$k$ and cognitive hierarchy models predict that players choose the same strategy in both IDENT and CS. (Note, this is true independent of the specific choice of level-0.) So, these models cannot explain the observed cross-treatment differences.

# 7 Discussion

**7.A Level-$k$ and Cognitive Hierarchy:** This paper is concerned with departures from RCBR, in the sense of $m$-rationality (page 15). There is a known connection between level-$k$ behavior and $m$-rationality: If a strategy is level-$k$ for $k \geq m$, then the strategy is consistent with $m$-rationality. (See, e.g., Crawford, Costa-Gomes, and Iriberri, 2013 or Schipper and Zhou, 2026. The same need not be true in the cognitive hierarchy model.) Importantly, the converse does not hold: In some games, all strategies are IU and, so, consistent with RCBR. Yet, the level-$k$ and cognitive hierarchy models—as implemented in practice—rule out certain IU strategies. Prominent examples include coordination games and the 11-20 games of Arad and Rubinstein (2012) and Alaoui and Penta (2016).[21] As a consequence, bounds identified through the lens of the level-$k$ or cognitive hierarchy model need not indicate bounds on reasoning about rationality. (This is true, even if the analyst is willing to maintain an assumption that the hierarchies are generated by a particular anchor. See Brandenburger, Friedenberg, and Kneeland, 2025.)

**7.B Ability to Engage in Interactive Reasoning** The introduction pointed to an important implication of our results: Departures from RCBR are not driven by limitations on players' ability to reason interactively. Arguably, this was an open question and addressing it is non-trivial. To understand these claims, let us point to other approaches one might have taken.

First, one might hope to address the question by directly eliciting subjects' hierarchies of beliefs. However, the very act of attempting to elicit the subjects' hierarchies may cause them to engage in higher levels of interactive reasoning than they may otherwise do. In turn, this can suggest evidence that departures from RCBR are not driven by limited ability to engage in interactive reasoning, even if they are.

Second, one might hope to address the question by varying "sophistication." There are two prominent versions of this approach. One varies whether a subject plays against a more vs. less "sophisticated" subject pool—e.g., graduate vs. undergraduate, game-theorist vs. granny, high vs. low Raven test score, computer vs. human, grandmaster vs. student, etc. (The technique was pioneered in Eichberger, Kelsey, and Schipper, 2008, Palacios-Huerta and Volij, 2009, Agranov, Potamites, Schotter, and Tergiman, 2012, and Georganas, Healy, and Weber, 2015.) A second version of the approach varies the training a given subject receives in "how to solve" the game, without varying the training of the pool of players the subject plays against. (This technique was pioneered by Alaoui, Janezic, and Penta, 2020.)

One interpretation of the approach is that a change in post-treatment behavior reflects

[21] There are other games where the conclusion is less clear, e.g., the beauty contest of Nagel (1995). See Seel and Tsakas (2017) and Section 8 in Brandenburger, Friedenberg, and Kneeland (2025).

a change in the subjects' ability to reason interactively. A second interpretation is that a change in post-treatment behavior reflects a change in how the subjects reason about rationality (i.e., their rationality bounds). Importantly, an observed change in post-treatment behavior cannot independently identify both the ability and rationality bounds. As a consequence, the post-treatment behavior is consistent with RCBR departures being driven/not driven by limited ability to reason interactively: If exactly one bound varies, then there must be a gap between the ability and rationality bounds. But, if both bounds exactly co-move, then there cannot be a gap between the ability and rationality bounds.

At the same time, in each of these approaches, it is not obvious that the treatment does, indeed, vary either of the subjects' bounds.[22] It may, instead, change subjects' beliefs, without changing their actual ability or rationality bounds. This is in the spirit of Eichberger, Kelsey, and Schipper's use of the approach and a lesson familiar from epistemic game theory: Subjects can have different beliefs about how different groups play the game, even if they engage in the same level of interactive reasoning about those groups. The converse also holds. A subject may engage in different levels of interactive reasoning or have different actual rationality bounds across groups, even if the subject has the same belief about play across the groups. Importantly, this lack of necessity or sufficiency is not only true in principle but it is also true for the experiments studied in the literature.

As an illustration, return to the 11-20 game. We noted the entire strategy set is IU. As such, under common belief of rationality, players may assign positive probability to any strategy. So, even if subjects engage in RCBR, their beliefs can be different for different subject pools. For instance, they may believe that students from different majors behave differently, based on different histories—e.g., different classroom lessons observed.[23]

**7.C Rationality vs. Irrationality** The identification strategy rests on an implicit assumption that observed behavior is a consequence of rational choice. An experimental literature has raised questions about the rationality hypothesis. These questions go back, at least, to Costa-Gomes and Weizsäcker (2008), who pointed out that subjects may not play a best response to stated beliefs. However, they also caution against concluding subjects must be irrational. In particular, they note: "It is possible for a subject to state a belief that differs from her underlying belief, the same way a subject might choose an action that she did not intend to play." (See page 757 in Costa-Gomes and Weizsäcker,

[22] In a different context, Fe, Gill, and Prowse (2021) argue that different measures of "sophistication" may be conceptually distinct and lead to different behavior.

[23] Arad and Rubinstein (2012) make a different point in the context of the 11-20 game—one that does not require varying sophistication. They argue that the 11-20 game is so simple that, in that game, there *must be* unlimited ability to engage in interactive reasoning. Even if that were prima facie obvious, their paper would not allow us to conclude that the bound on ability is strictly higher than the identified rationality bound—after all, in the 11-20 game, all behavior is consistent with RCBR.

2008.) That is, behavior that appears irrational may, in fact, reflect noisy decision making or measurement errors (discussed in Section 6.1.)

At the same time, the conclusion of irrationality is not inevitable. Healy points out that, even within a given experimental design, the conclusion of irrationality is "highly game-specific." (See page 4 of Healy, 2024.) In fact, the results in Healy suggest that departures from rationality are unlikely in settings like Figures 2.1-2.2.

**7.D Focal Beliefs Model** Sprenger and Zhao (2021) propose an alternate model of strategic reasoning, the focal beliefs model. That model is related to but distinct from the HB model. In particular, the focal beliefs model can be reinterpreted as follows: A player either assigns probability 1 to rationality or probability 1 to the other player adopting a heuristic that is based on maximizing the sum (subject to a no-zero requirement). Importantly, the focal beliefs model is inconsistent with our data. It precludes *any* non-constant non-IU play and it predicts the same distribution of play across IDENT and CS. Moreover, the HB model better explains their data. In particular, 40% of their subjects play $(\mathrm{a}, \mathrm{c}_*)$ in the role P2. This non-constant play is inconsistent with the focal beliefs model but consistent with the HB model. (See Figure 5.2b.)

**7.E The Portability Puzzle** The level-$k$ and cognitive hierarchy literatures point to a peculiar feature of the data: "Levels of reasoning," as defined in those models, is not a portable concept. (See, e.g., Georganas, Healy, and Weber, 2015.) That is, subjects who are characterized as being level-$k$ reasoners in the context of one game may be characterized as being level-$j$ reasoners in the context of another game, for some $j \neq k$. One might view this as unusual, expecting subjects to reason similarly across similar games.

We can understand this very same behavior in light of the SB and HB models. A subject's identified rationality bound may change across games, even if her actual rationality bound does not. For instance, consider a subject that has a strategic bound of $k \geq 2$. Suppose further that the subject assigns probability $p \in (0, 1)$ to the other subject's rationality and $(1 - p)$ to the other subject adopting a heuristic. Then small changes in payoffs can lead to changes in the subject's identified rationality bound, even if her actual strategic and rationality bounds do not vary across treatments. That is, the subject's reasoning may well be portable. Yet, the level-$k$ and cognitive hierarchy literatures may well view this same behavior as reflecting different "levels of reasoning."

This raises a question: Are the actual rationality and strategic bounds a portable concept? We leave this for future research.

**7.F A Route to New Solution Concepts** The experimental results show that there may be systematic ways in which strategic reasoning departs from reasoning about rationality. The HB model is one candidate and it is supported by the data. But, out of intellectual cautiousness, we stop short of offering the model as a novel solution concept.

There are many important questions about the nature of strategic reasoning that are simply not addressed by the HB model. For instance, we designed the experiments so that a class of heuristics coincide; the experiment cannot speak to whether a specific heuristic within that class drives heuristic reasoning. Likewise, we took no stand on how players reason across games that differ by more than a relabeling of actions. (For that reason, we took no stand on how players reason across player roles.) These—and, no doubt, other considerations—deserve further analyses. We see the results here as opening up a literature.

## References


Agranov, M. and P. Ortoleva (2017). Stochastic choice and preferences for randomization. *Journal of Political Economy 125*(1), 40–68.

Agranov, M., E. Potamites, A. Schotter, and C. Tergiman (2012). Beliefs and endogenous cognitive levels: An experimental study. *Games and Economic Behavior 75*, 449–463.

Alaoui, L., K. A. Janezic, and A. Penta (2020). Reasoning about others' reasoning. *Journal of Economic Theory 189*, 105091.

Alaoui, L. and A. Penta (2016). Endogenous depth of reasoning. *The Review of Economic Studies 83*(4), 1297–1333.

Arad, A. and A. Rubinstein (2012). The 11-20 money request game: A level-$k$ reasoning study. *American Economic Review 102*, 3561–3573.

Battigalli, P., A. Friedenberg, and M. Siniscalchi (2021). Epistemic game theory: Reasoning about strategic uncertainty. unpublished.

Battigalli, P. and M. Siniscalchi (2002). Strong belief and forward induction reasoning. *Journal of Economic Theory 106*(2), 356–391.

Becker, G. M., M. H. DeGroot, and J. Marschak (1963). Stochastic models of choice behavior. *Behavioral Science 8*(1), 41–55.

Block, D. and J. Marschak (1960). Random orderings and stochastic theories of responses. In I. Olkin et al. (Eds.), *Contributions to Probability and Statistics*. Stanford University Press.

Brandenburger, A. (2007). The power of paradox: some recent developments in interactive epistemology. *International Journal of Game Theory 35*(4), 465–492.

Brandenburger, A., A. Danieli, and A. Friedenberg (2021). The implications of finite-order reasoning. *Theoretical Economics 16*(4), 1605–1654.

Brandenburger, A., A. Friedenberg, and T. Kneeland (2025). From anchors to hierarchies: Identifying levels of reasoning in games. Technical report.

Brocas, I., J. D. Carrillo, and A. Sachdeva (2018). The path to equilibrium in sequential and simultaneous games: A mousetracking study. *Journal of Economic Theory 178*, 246–274.

Camerer, C. F., T.-H. Ho, and J.-K. Chong (2004). A cognitive hierarchy model of games. *Quarterly Journal of Economics 119*(3), 861–898.

Chen, D. L., M. Schonger, and C. Wickens (2016). otree—an open-source platform for laboratory, online, and field experiments. *Journal of Behavioral and Experimental Finance 9*, 88–97.

Costa-Gomes, M., V. P. Crawford, and B. Broseta (2001). Cognition and behavior in normal-form games: An experimental study. *Econometrica 69*, 1193–1235.

Costa-Gomes, M. A. and V. P. Crawford (2006). Cognition and behavior in two-person guessing games: An experimental study. *American Economic Review 96*, 1737–1768.

Costa-Gomes, M. A. and G. Weizsäcker (2008). Stated beliefs and play in normal-form games. *The Review of Economic Studies 75*(3), 729–762.

Crawford, V. P., M. A. Costa-Gomes, and N. Iriberri (2013). Structural models of nonequilibrium strategic thinking: Theory, evidence, and applications. *Journal of Economic Literature 51*, 5–62.

Dekel, E. and M. Siniscalchi (2014). Epistemic game theory. *Handbook of Game Theory 4*.

Dwenger, N., D. Kübler, and G. Weizsäcker (2018). Flipping a coin: Evidence from university applications. *Journal of Public Economics 167*, 240–250.

Eichberger, J., D. Kelsey, and B. C. Schipper (2008). Granny versus game theorist: Ambiguity in experimental games. *Theory and Decision 64*, 333–362.

Fe, E., D. Gill, and V. Prowse (2021). Cognitive skills, strategic sophistication, and life outcomes. https://web.ics.purdue.edu/~gill53/FeGillProwse_CogSkills.pdf.

Georganas, S., P. J. Healy, and R. A. Weber (2015). On the persistence of strategic sophistication. *Journal of Economic Theory 159*, 369–400.

Ghosh, S., A. Heifetz, and R. Verbrugge (2015). Do players reason by forward induction in dynamic perfect information games? In *Proceedings of the 15th conference on Theoretical Aspects of Rationality and Knowledge (TARK)*.

Ghosh, S. and R. Verbrugge (2018). Studying strategies and types of players: Experiments, logics and cognitive models. *Synthese 195*, 4265–4307.

Gillen, B., E. Snowberg, and L. Yariv (2019). Experimenting with measurement error: Techniques with applications to the Caltech cohort study. *Journal of Political Economy 127*(4), 1826–1863.

Greiner, B. (2015). Subject pool recruitment procedures: organizing experiments with orsee. *Journal of the Economic Science Association 1*(1), 114–125.

Gul, F. and W. Pesendorfer (2006). Random expected utility. *Econometrica 74*(1), 121–146.

Harless, D. W. and C. F. Camerer (1994). The predictive utility of generalized expected utility theories. *Econometrica 62*(6), 1251–1289.

Healy, P. J. (2024). Epistemic experiments: Utilities, beliefs, and irrational play. https://healy.econ.ohio-state.edu/papers/Healy-EpistemicExperiments.pdf.

Kneeland, T. (2015). Identifying higher-order rationality. *Econometrica 83*, 2065–2079.

Li, Y. X. and B. C. Schipper (2020). Strategic reasoning in persuasion games: An experiment. *Games and Economic Behavior 121*, 329–367.

McFadden, D. (1974). Conditional logit analysis of qualitative choice behavior. In P. Zarembka (Ed.), *Frontiers in Econometrics*, pp. 105–142. New York: Academic Press.

Nagel, R. (1995). Unraveling in guessing games: An experimental study. *American Economic Review 85*, 1313–1326.

Palacios-Huerta, I. and O. Volij (2009). Field centipedes. *American Economic Review 99*(4), 1619–35.

Schipper, B. C. and H. Zhou (2026). Level-k thinking in the extensive form. *Economic Theory 81*(1), 149–189.

Seel, C. and E. Tsakas (2017). Rationalizability and nash equilibria in guessing games. *Games and Economic Behavior 106*, 75–88.

Siniscalchi, M. (2022). Structural rationality in dynamic games. *Econometrica 90*(5), 2437–2469.

Sprenger, C. and W. Zhao (2021). Predictive accuracy for measures of higher order rationality - an experiment using ring game.

Stahl, D. O. and P. W. Wilson (1994). Experimental evidence on players' models of other players. *Journal of economic behavior & organization 25*(3), 309–327.

Stahl, D. O. and P. W. Wilson (1995). On players' models of other players: Theory and experimental evidence. *Games and Economic Behavior 10*, 218–254.

Tan, T. C.-C. and S. R. da Costa Werlang (1988). The bayesian foundations of solution concepts of games. *Journal of Economic Theory 45*(2), 370–391.

Thurstone, L. L. (1927). A law of comparative judgment. *Psychological Review 34*, 273–286.

# Supplemental Appendix for "Beyond Reasoning about Rationality: Evidence of Strategic Reasoning"

Amanda Friedenberg* Terri Kneeland†

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## Appendix A The Identification Strategy

It will be convenient to introduce notation that will be used throughout the appendices. Given a (compact metric) set $Y$, write $\Delta(Y)$ for the set of probability measures on $Y$.

Denote the set of strategies by $S = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\} \times \{\mathrm{a}_*, \mathrm{b}_*, \mathrm{c}_*, \mathrm{d}_*\}$ and the set of observations by $X = S \times S \times S \times S$. As in Section 5 of the main text, we will write

$$\mathrm{Inv}_i = \{(\mathrm{a}, \Pi_i(\mathrm{a})), (\mathrm{b}, \Pi_i(\mathrm{b})), (\mathrm{c}, \Pi_i(\mathrm{c})), (\mathrm{d}, \Pi_i(\mathrm{d}))\}$$

Write $\mathcal{SB}^k$ for the **set of observations with an identified strategic bound of** $k$. So, when $k \geq 2$, an observation $x = (x(1), x(2), x(3), x(4)) \in \mathcal{SB}^k$ if and only if (i) $x(1) = (\mathrm{a}, \mathrm{d}_*)$; (ii) $x(2)$ is a unique best response under a 2-strategic belief; and (iii) for each $i \in \{j : 4 \geq j > k\}$, $x(i)$ is constant.

### A.1 Identification of the Strategic Bounds

In this appendix, we draw a connection between behavior of a rational subject that has a strategic bound of $k$ and an observation $x \in \mathcal{SB}^k$. In doing so, we equate a *rational* subject with a subject that plays a unique best response given her first-order belief (i.e., belief about observations). (This fits with the earlier assumption that a rational subject is not indifferent between any two strategies.) We will show that an observation $x \in \mathcal{SB}^k$ if and only if (i) a rational subject with a strategic bound of $k$ would play $x$, and (ii) a rational subject with a strategic bound of $j < k$ would not play $x$.

**Proposition A.1.** *For each $k \geq 1$ the following hold:*

*(i) If a rational subject with a strategic bound of $k$ chooses an observation $x$, then $x \in \mathcal{SB}^j$ for some $j \leq k$.*

*University of Michigan, amanda@amandafriedenberg.org, http://amandafriedenberg.org
†University of Calgary, terri.kneeland@ucalgary.ca, http://tkneeland.com

*(ii) If $x \in \mathcal{SB}^k$, then there exists a rational subject with a strategic bound of $k$ that chooses $x$.*

Proposition A.1 serves to characterize the behavior of a rational subject who has a strategic bound of $k$. Part (i) argues that a necessary condition is that the behavior has an identified strategic bound of $j \leq k$. Part (ii) gives sufficiency—an observation with an identified strategic bound of $k$ must be consistent with the behavior of a rational subject who has a strategic bound of $k$.

The result is immediate for $k = 1$. So we focus on showing the result for $k \geq 2$.

**Proof of Necessity** Proposition A.1(i) will follow immediately from the following Lemmata:

**Lemma A.1.** *Consider a rational subject who has a strategic bound of $k \geq 2$. In the role of P2, the subject plays a strategy that is a unique best response to a 2-strategic belief.*

**Lemma A.2.** *Fix some $j, k$ with $4 \geq j > k \geq 2$. Consider a rational subject who has a strategic bound of $k$. In the role of P$j$, the subject plays a constant strategy.*

To show these results, it will be useful to, in turn, have two auxiliary lemmata.

**Lemma A.3.** *Let P$i$ = P1, P3, P4. If P$i$ is strategic and believes that P$(i-1)$ has a strategic bound of 1, then P$i$ has a constant belief.*

*Proof.* Suppose P$i$ believes that P$(i-1)$ has a strategic bound of 1. Then, P$i$ believes that "P$(i-1)$ is strategic and satisfies the Principle of Non-Strategic Reasoning." By the Principle of Strategic Reasoning, P$i$'s belief satisfies $\Pr_i(\mathrm{d}, \mathrm{e}_*) > 0$ only if $\Pi_{(i-1)}(\mathrm{d}) = \mathrm{e}_*$. Since $i - 1 \neq 1$, P$i$ has a constant belief. □

**Lemma A.4.** *If P1 is strategic and believes that P4 has a strategic bound of 2, then P1 has a constant belief.*

*Proof.* Suppose P1 believes that P4 has a strategic bound of 2. Then, P1 believes that "P4 is 2-strategic and believes that P3 has a strategic bound of 1." By Lemma A.3, P1 believes that "P4 is 1-strategic and has a constant belief." Applying the Principle of Strategic Reasoning, P1 believes that P4 plays a constant strategy. □

*Proof of Lemma A.1.* Since the subject has a strategic bound of $k \geq 2$, in the role of P2, the subject satisfies the Principle of Strategic Reasoning. It suffices to show that P2 believes P1 has a constant belief: If so, the Principle of Strategic Reasoning implies that P2's belief assigns probability 1 to

$$\mathrm{Inv}_1 = \{(\mathrm{a}, \Pi_1(\mathrm{a})), (\mathrm{b}, \Pi_1(\mathrm{b})), (\mathrm{c}, \Pi_1(\mathrm{c})), (\mathrm{d}, \Pi_1(\mathrm{d}))\},$$

i.e., P2's belief is a 2-strategic belief.

For $k = 2$, this follows immediately from the fact that the subject believes that the player in the role of P1 satisfies the Principle of Non-Strategic Reasoning. For $k = 3$ (resp. $k = 4$), P2 believes that P1 has a strategic bound of 2 (resp. 3). As such, P2 believes that "P1 believes P4 has a strategic bound of 1 (resp. 2)." So, by Lemma A.3 (resp. Lemma A.4), P2 believes that P1 has a constant belief. □

*Proof of Lemma A.2.* Since the subject has a strategic bound of $k \geq 2$, the subject satisfies the Principle of Strategic Reasoning. It suffices to show that, if $j > k$, P$j$ believes P$(j-1)$ has a constant belief: If so, the Principle of Strategic Reasoning implies that P$j$'s belief assigns probability 1 to

$$\text{Inv}_{j-1} = \{(\text{a}, \Pi_{j-1}(\text{a})), (\text{b}, \Pi_{j-1}(\text{b})), (\text{c}, \Pi_{j-1}(\text{c})), (\text{d}, \Pi_{j-1}(\text{d}))\}.$$

Note, P$j$'s belief is then a constant belief, as $j-1$ is either 2 or 3. Since a rational subject has a unique best response to her belief, the subject plays a constant strategy.

Since $4 \geq j > k$, $k$ is either 2 or 3. If $k = 2$, the claim follows immediately from the fact that the subject believes the player in the role of P$(j-1)$ satisfies the Principle of Non-Strategic Reasoning. If $k = 3$, P$j$ believes that P$(j-1)$ has a strategic bound of 2. As such, P$j$ believes that "P$(j-1)$ believes that P$(j-2)$ has a strategic bound of 1 (resp. 2)." So, by Lemma A.3, P$j$ believes that P$(j-1)$ has a constant belief. □

**Proof of Sufficiency** To show Proposition A.1(ii), we need to lay out certain epistemic formalisms. We use the minimal formalism necessary, introducing first-, second-, and higher-order beliefs, as they are needed. We don't redefine our concepts within those formalisms; we will be careful with words in a way that we think is clear.

Two factors are important in how we define these higher-order beliefs. First, we think of a subject as having hierarchies of beliefs about the observed behavior of a *single* other subject. This is for notational simplicity. Second, beliefs are defined over observations (instead of specifying beliefs in each player role about strategies played). This is because we will want to think about a subject with a heuristic bound of 2 believing that the other subject has the same belief independent of the player role.

It is convenient to define **first-order belief maps** for observations with an identified strategic bound of $k \geq 2$. Note, a first-order belief is a belief on $X^1 = X$.

**Definition A.1.** *Fix $k \geq 2$. For each $i$, define a map $f_i^k : \mathcal{SB}^k \to \Delta(S)$ so that, for each $x = (x(1), x(2), x(3), x(4)) \in \mathcal{SB}^k$, the following hold:*

- *If $i = 2$, $f_i^k(x)$ is a 2-strategic belief under which $x(2)$ is a unique best response.*

- *If* $i \in \{1\} \cup \{j : j > k\}$, $f_i^k(x)$ *is a constant belief under which* $x(i)$ *is a unique best response.*
- *If* $i \in \{j : k \geq j > 2\}$, $f_i^k(x)$ *is a belief under which* $x(i)$ *is a unique best response.*

*The* first-order belief map for $k$ *is a map* $f^k : \mathcal{SB}^k \to \Delta(X)$ *where, for each observation* $x = (x(1), x(2), x(3), x(4)) \in \mathcal{SB}^k$, $f^k(x) = f_2^k(x) \otimes f_3^k(x) \otimes f_4^k(x) \otimes f_1^k(x)$.[1]

A second-order belief is a belief in $X^2 = X^1 \times \Delta(X^1)$. To simplify notation, we think of a third-order belief as a belief about $X^3 = X^1 \times \Delta(X^2)$—i.e., specifying a belief about observations and other subjects' beliefs about "observations and first-order beliefs." (We can read off the second-order belief via marginalization.) And, similarly, we think of a fourth-order belief as a belief about $X^4 = X^1 \times \Delta(X^3)$.

It will suffice to focus on degenerate higher-order beliefs. Thus, given a belief $\nu \in \Delta(X^k)$, we will write $\delta_\nu$ to indicate the belief in $\Delta(\Delta(X^k))$ that assigns probability 1 to $\nu$. To construct the relevant degenerate beliefs, it is convenient to fix two objects: First, let $\Pr \in \Delta(S)$ be a constant belief and let $\overrightarrow{\Pr} \in \Delta(X)$ be the associated product measure $(\Pr \otimes \Pr \otimes \Pr \otimes \Pr)$. Second, for each $k \in \{2, 3\}$, fix $y^k \in \mathcal{SB}^k$.

Now, inductively define maps $h^k : \mathcal{SB}^k \to \Delta(X^k)$.

- For each $x \in \mathcal{SB}^2$, $h^2(x) = f^2(x) \otimes \delta_{\overrightarrow{\Pr}}$.
- Assume $h^k$ has been defined for $k \in \{2, 3\}$. For each $x \in \mathcal{SB}^{k+1}$, let $h^{k+1}(x) = f^{k+1}(x) \otimes \delta_{h^k(y^k)}$.

The following Lemma establishes Proposition A.1(ii).

**Lemma A.5.** *Let* $k \geq 2$. *For each observation* $x \in \mathcal{SB}^k$,

*(i)* $x$ *is a unique best response under* $f^k(x)$, *and*

*(ii)* $(x, h^k(x))$ *has a strategic bound of* $k$.

*Proof.* By construction, each $x \in \bigcup_{k \geq 2} \mathcal{SB}^k$ is a unique best response under $f^k(x)$. So, we focus on showing that each $(x, h^k(x))$ has a strategic bound of $k$.

$k = 2$: Fix $x = (x(1), x(2), x(3), x(4)) \in \mathcal{SB}^2$. First observe that, for each $i$ with $f_i^2(x)$ constant, $x(i) \in \text{Inv}_i$. Thus, $(x, h^2(x))$ is 1-strategic. Moreover, by construction, each $f_i^2(x)$ assigns probability 1 to $\text{Inv}_{i-1}$. Thus, $(x, h^2(x))$ is 2-strategic. Moreover, since $h^2(x)$ assigns probability 1 to the other player having the same constant belief Pr across player

[1] We adopt this ordering so that $f^k$ is indeed a distribution on observations $(y(1), y(2), y(3), y(4))$.

roles, $(x, h^2(x))$ also satisfies the Principle of Non-Strategic Reasoning. Thus, $(x, h^2(x))$ has a heuristic bound of 2.

$k \geq 3$: Assume the result holds for $k \in \{2, 3\}$. Fix some $x = (x(1), x(2), x(3), x(4)) \in \mathcal{SB}^{k+1}$. First observe that, for each $i$ with $f_i^{k+1}(x)$ constant, $x(i) \in \text{Inv}_i$. Moreover, $h^{k+1}(x)$ assigns probability 1 to some $k^{th}$-order belief $h^k(y^k)$, where $y \in \mathcal{SB}^k$. So, by the induction hypothesis, $(x, h^{k+1}(x))$ has a heuristic bound of $k+1$. □

### A.2 Identifying the Gap

**Lemma A.6.** *If an observation has an identified strategic bound of $k$, then the identified rationality bound must be $m \leq k$.*

*Proof.* Fix an observation $(x(1), x(2), x(3), x(4)) \in \mathcal{SB}^k$. Suppose, contra hypothesis, that the observation has an identified rationality bound of $m > k$. Then, $x(k+1)$ is the non-constant IU strategy, contradicting $(x(1), x(2), x(3), x(4)) \in \mathcal{SB}^k$. □

**Lemma A.7.** *Fix a rational subject who has a strategic bound of $k$.*

*(i) If the subject's behavior generates an observation with an identified rationality bound of $m$, then $m \leq k$.*

*(ii) If the subject's behavior generates an observation with an identified gap, then the subject's rationality bound must be strictly lower than $k$.*

*Proof.* Since the subject is rational and has a strategic bound of $k$, the subject's behavior generates an observation with an identified strategic bound of $j \leq k$. (See Proposition A.1(i).) By Lemma A.6, the identified rationality bound must be $m \leq j$. Thus, $m \leq k$, establishing part (i).

Now observe that, if the subject's behavior generates an identified gap, then the subject's identified rationality bound must be $m < 4$. As a consequence, the subject's rationality bound is some $n \leq m$. Put together, $n \leq m < j \leq k$, establishing part (ii). □

## Appendix B Heuristic Beliefs Model

### B.1 Heuristic Beliefs vs. Strategic Beliefs

**Lemma B.1.** *If a subject is $k$-heuristic, the subject is $k$-strategic.*

*Proof.* For $k = 1$, the claim is by definition. Fix a subject that is 2-heuristic. Then the subject is 1-strategic and, in each player role P$i$, the subject has simple strategic belief $\text{Pr}_i$.

So, if $\Pr_i(\mathrm{e}, \mathrm{f}_*) > 0$, then $(\mathrm{e}, \mathrm{f}_*)$ is either rational or invariant. As such, to show that the subject satisfies the Principle of Strategic Reasoning, it suffices to show that, if P$i$ believes that P$(i-1)$ has a constant belief and $\Pr_i(\mathrm{e}, \mathrm{f}_*) > 0$ for a rational $(\mathrm{e}, \mathrm{f}_*)$, then $(\mathrm{e}, \mathrm{f}_*)$ is invariant. This follows from the fact that P$i$ believes that, if P$(i-1)$ is rational, then he is not indifferent between any two strategies.

Now suppose the claim holds for $k \in \{2, 3\}$. Fix a $(k+1)$-heuristic subject. By the induction hypothesis, the subject is $k$-strategic and believes that others are $k$-strategic. Thus, the subject is $(k+1)$-strategic. □

**Lemma B.2.** *If a subject has a heuristic bound of $k$, the subject has a strategic bound of $k$.*

*Proof.* For $k = 1$, the claim is by definition. Assume the claim holds for $k \geq 1$, where $3 \geq k$. Fix a subject with a heuristic bound of $(k+1)$. By Lemma B.1, the subject is $(k+1)$-strategic. Moreover, by the induction hypothesis, the subject believes that the others have a strategic bound of $k$. Thus, the subject also has a strategic bound of $(k+1)$. □

**Lemma B.3.** *If an observation is identified as having a heuristic bound of $k$, it is identified as having a strategic bound of $k$.*

*Proof.* An observation is identified as having a heuristic bound of 1 if and only if it is identified as having a strategic bound of 1. As a consequence, if an observation is identified as having a heuristic bound of 2, it is also identified as having a strategic bound of 2. If it is identified as having a strategic bound of 2 but not identified as having a heuristic bound of 2, it must be that the behavior in the role of P2 is not a unique best response under a 2-heuristic belief. As a consequence, the observation won't be identified as having a heuristic bound of 3 or 4. So, if the observation is identified as having a heuristic bound of 3, it must be identified as having a strategic bound of 3. Repeating the argument gives the conclusion for $k = 3$ and $k = 4$. □

## B.2 Identification of Heuristic Bounds

Let $\mathcal{HB}^k$ be **the set of observations with an identified heuristic bound of** $k$. So, $x \in \mathcal{HB}^1$ if and only if $x \in \mathcal{SB}^1$. Moreover, for $k \geq 2$, $\mathcal{HB}^k = \prod_{i \leq k} \mathrm{HB}_i \times \prod_{i>k} \mathrm{Inv}_i$.

To define the non-indifference condition, it is convenient to introduce the following definition: Say P$i$ is **1-rational** if she plays a best response given a belief about P$(i-1)$'s play of the game.[2] Say P$i$ is **$m$-rational** if she plays a best response given a belief that assigns probability 1 to the event that "P$(i-1)$ is $(m-1)$-rational."

[2]Notice, here we do not include a requirement that the best response is unique.

Say a subject satisfies **first-order belief of non-indifference** if, in each player role P$i$, she believes:

"If P$(i-1)$ is 1-rational, he is not indifferent between any two strategies."

Inductively, a subject satisfies **$k^{th}$-order belief of non-indifference** if, in each player role P$i$, she believes:

"If P$(i-1)$ is $(k-1)$-rational, he satisfies $(k-1)^{th}$-order belief of non-indifference."

In what follows, we assume that, if a subject is $k$-heuristic, the subject satisfies $(k-1)^{\text{th}}$-order belief of non-indifference. If this holds for all $k \in \{1,2,3,4\}$, we say the **belief of non-indifference assumption** holds.

**Proposition B.1.** *Fix $k \geq 2$.*

*(i) Suppose that the belief of non-indifference assumption holds. If a rational subject with a heuristic bound of $k$ chooses an observation $x$, then $x \in \mathcal{HB}^j$ for some $j \leq k$.*

*(ii) If $x \in \mathcal{HB}^k$, then there exists a rational subject with a heuristic bound of $k$ that both satisfies belief of non-indifference and chooses $x$.*

Proposition B.1 characterizes the behavior of a rational subject with a heuristic bound of $k$. Part (i) says that a necessary condition is that the subject's behavior has an identified heuristic bound of $j \leq k$. Part (ii) establishes the converse. Any observation with an identified heuristic bound of $k$ can be generated by the behavior of a rational subject with a heuristic bound of $k$.

**Proof of Necessity** To prove part (i), we make use of the following Lemma.

**Lemma B.4.** *Suppose that the belief of non-indifference assumption holds. Fix $k \geq j \geq 2$. If a rational subject is $k$-heuristic then, in the role of P$j$, the subject plays some strategy $x(j) \in \text{HB}_j$.*

*Proof.* The proof is by induction on $j$. For $j = 2$, the proof is immediate from the fact that $k$-heuristic implies 2-heuristic and no subject is indifferent between any two actions. So suppose the claim holds for $j \in \{2,3\}$ and let $k \geq (j+1)$. Note, a $k$-heuristic subject satisfies $j^{\text{th}}$-order belief of non-indifference and believes that others are $j$-heuristic. So, by the induction hypothesis, a $k$-heuristic subject believes that the rational strategies, in the role of P$j$, are the strategies in $\text{HB}_j$. Since $\text{Inv}_j$ is the set of constant strategies, a $k$-heuristic subject has a $(j+1)$-heuristic belief in the role of P$(j+1)$. Since no subject is indifferent between any two actions, a $k$-heuristic subject plays some strategy $x(j+1)$ that is a unique best response to a $(j+1)$-heuristic belief. □

*Proof of Proposition B.1(i).* Immediate from Lemma B.4, Lemma B.2, and Lemma A.2. □

**Proof of Sufficiency** To show Proposition B.1(ii), we need, once again, to introduce hierarchies of beliefs. We follow the approach in Appendix A.1, making suitable modifications. We often use the same notation for adjacent concepts. We do this to draw parallels. No confusion should result.

**Definition B.1.** *Fix $k \geq 2$. For each $i$, define a map $f_i^k : \mathcal{HB}^k \to \Delta(S)$ so that, for each $x = (x(1), x(2), x(3), x(4)) \in \mathcal{HB}^k$, the following hold:*

- *If $i \in \{2, \ldots, k\}$, $f_i^k(x)$ is a $i$-heuristic belief under which $x(i)$ is a unique best response.*
- *If $i \in \{1\} \cup \{j > k\}$, $f_i^k(x)$ is constant ordered-heuristic belief under which $x(i)$ is a unique best response.*

*The **first-order belief map for** $k$ is a map $f^k : \mathcal{HB}^k \to \Delta(X)$ where, for each observation $x = (x(1), x(2), x(3), x(4)) \in \mathcal{HB}^k$, $f^k(x) = f_2^k(x) \otimes f_3^k(x) \otimes f_4^k(x) \otimes f_1^k(x)$.*

Note, for each $k$ and each $x \in \mathcal{HB}^k$, $f^k(x)$ is a distribution on $X$. For each $k$, we can provide a uniform bound on the supports of distributions $f^k(x)$. In particular, set

$$\text{BSupp}^k = \begin{cases} \text{Inv}_1 \times \text{Inv}_2 \times \text{Inv}_3 \times \text{Inv}_4 & \text{if } k = 2 \\ \text{Inv}_1 \times (\text{HB}_2 \cup \text{Inv}_2) \times \text{Inv}_3 \times \text{Inv}_4 & \text{if } k = 3 \\ \text{Inv}_1 \times (\text{HB}_2 \cup \text{Inv}_2) \times (\text{HB}_3 \cup \text{Inv}_3) \times \text{Inv}_4 & \text{if } k = 4. \end{cases}$$

For each $k \geq 2$ and each $x \in \mathcal{HB}^k$, $\text{Supp} f^k(x) \subseteq \text{BSupp}^k$. Moreover, for each $k \geq 2$, $\mathcal{HB}^{k-1} \subseteq \text{BSupp}^k$.

With this in mind, for each $k \geq 2$, construct a mapping $g^k : \text{BSupp}^k \to \mathcal{HB}^{k-1}$ that satisfies the following two properties: First, for each $x \in \mathcal{HB}^{k-1} \subseteq \text{BSupp}^k$, $g^k(x) = x$. Second, for each $x = (x(1), x(2), x(3), x(4)) \in \text{BSupp}^k \backslash \mathcal{HB}^{k-1}$, $g^k(x) = (y(1), y(2), y(3), y(4))$ where $y(i) = x(i)$ if $x(i) \in \text{HB}_i$. (Observe that this can be done, since $\mathcal{HB}^{k-1} = \prod_{i<k} \text{HB}_i \times \prod_{i \geq k} \text{Inv}_i$.)

We also fix a constant belief $\text{Pr} \in \Delta(S)$ under which each P$i$ has a unique best response. Let $\overrightarrow{\text{Pr}} \in \Delta(X)$ be the associated product measure $(\text{Pr} \otimes \text{Pr} \otimes \text{Pr} \otimes \text{Pr})$.

Now, inductively define maps $h^k : \mathcal{HB}^k \to \Delta(X^k)$ as follows. First, for each $x \in \mathcal{HB}^2$, $h^2(x) = f^2(x) \otimes \delta_{\overrightarrow{\text{Pr}}}$. Assume that $h^k : \mathcal{HB}^k \to \Delta(X^k)$ has been defined for $k \in \{2, 3\}$. For each $x \in \mathcal{HB}^{k+1}$, let $h^{k+1}(x) = \mu \in \Delta(X \times \Delta(X^k))$ that satisfies the following:

$$\mu(y, \nu) = \begin{cases} f^{k+1}(x)(y) \times \delta_\nu & \text{if } y \in \text{Supp} f^{k+1}(x) \text{ and } \nu = h^k(g^{k+1}(y)) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f^{k+1}(x) \in \Delta(X \times \Delta(X^k))$.

**Lemma B.5.** *Fix $k \geq 2$. For each $x \in \mathcal{HB}^k$, $(x, h^k(x))$ is rational and has a heuristic bound of $k$.*

*Proof.* Begin by fixing $x = (x(1), x(2), x(3), x(4)) \in \mathcal{HB}^2$. First observe that each $x(i)$ is a unique best response to the invariant belief $f_i^2(x)$. Thus, $(x, h^2(x))$ is rational and 1-heuristic. Moreover, each $f_i^2(x)$ is an ordered-heuristic belief. And, since $h^2(x)$ assigns probability 1 to the other playing Pr independent of the player role, $h^2(x)$ believes that, if the other subject plays a best response, that best response is unique. This establishes that $(x, h^2(x))$ is 2-heuristic. Finally, since $h^2(x)$ assigns probability 1 to the other player having the same constant belief Pr across player roles, $(x, h^2(x))$ also satisfies the Principle of Non-Strategic Reasoning. Thus, $(x, h^2(x))$ has a heuristic bound of 2.

Now assume the claim holds for $k \in \{2, 3\}$ and fix some $x \in \mathcal{HB}^{k+1}$. Note that $x$ is a unique best response under $f^{k+1}(x)$. So, $(x, h^{k+1}(x))$ is rational. With this, we focus on showing that $(x, h^{k+1}(x))$ has a heuristic bound of $k$.

Begin by observing that, if $f_i^{k+1}(x)$ is a constant belief, $x(i)$ must be invariant. (This uses the fact that $x$ is a unique best response under $f^{k+1}(x)$.) So, $(x, h^{k+1}(x))$ is 1-heuristic.

Now, let $\mu \equiv h^{k+1}(x)$. It suffices to show that, if $\mu(y, \nu) > 0$ then $(y, \nu)$ has a heuristic bound of $k$. Since $\mu(y, \nu) > 0$, $y = (y(1), y(2), y(3), y(4)) \in \text{Supp} f^{k+1}(x)$ and $\nu = h^k(g^{k+1}(y))$. If $y \in \mathcal{HB}^k$, the result follows from the induction hypothesis. If $y \notin \mathcal{HB}^k$, then there exists $z = (z(1), z(2), z(3), z(4))$ so that $g^k(y) = z$. Note, $(z, h^k(z))$ is rational and has a heuristic bound of $k$. (This follows from the induction hypothesis.) We use this fact to show that $(y, h^k(z))$ has a heuristic bound of $k$.

First observe that $(y, h^k(z))$ must be 1-heuristic. To see this, note that $y(1), y(3), y(4)$ are invariant and $y(2) = z(2)$. So, using the fact that $(z, h^k(z))$ is 1-heuristic, $(y, h^k(z))$ must also be 1-heuristic. From this and the fact that $(z, h^k(z))$ has a heuristic bound of $k$, it follows that $(y, h^k(z))$ also has a heuristic bound of $k$. □

Let $S(i)$ be the set of strategies in the role of P$i$. A basic induction argument establishes the following:

**Lemma B.6.** *Fix $k \geq 2$ and $x \in \mathcal{HB}^k$. If $h^k(x)$ assigns strictly positive probability to $(y, \nu) \in X \times \Delta(X^{k-1})$ and $(y, \nu)$ is $(k-1)$-rational in the role of Pi, then $y(i)$ is a strict best response under $f_i^{k-1}(y) = \text{marg}_{S(i)}\nu$.*

*Proof of Proposition B.1(ii) .* Immediate from Lemmata B.5 and B.6. □

# Appendix C Additional Tables and Figures

| P1 | P2 | P3 | P4 |
|---|---|---|---|
| $(a, d_*)$ | $(a, a_*)$ | $(a, a_*)$, $(d, d_*)$ | $(b, b_*)$, $(c, c_*)$,$(d, d_*)$ |
| $(a, d_*)$ | $(a, a_*)$ | $(b, b_*)$ | $(b, b_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(b, b_*)$ | $(a, a_*)$, $(b, b_*)$ | $(b, b_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(c, c_*)$ | $(a, a_*)$ | $(b, b_*)$, $(c, c_*)$ |
| $(a, d_*)$ | $(c, c_*)$ | $(b, b_*)$ | $(b, b_*)$ |
| $(a, d_*)$ | $(c, c_*)$ | $(c, c_*)$ | $(c, c_*)$ |
| $(a, d_*)$ | $(c, c_*)$,$(d, d_*)$ | $(d, d_*)$ | $(b, b_*)$, $(c, c_*)$ |

Table C.1. Observations Identified as Strategic/Heuristic Bound of 1
For Both IDENT and CS

| P1 | P2 | P3 | P4 |
|---|---|---|---|
| $(a, d_*)$ | $(a, a_*)$ | $(a, a_*)$, $(d, d_*)$ | $(a, a_*)$ |
| $(a, d_*)$ | $(a, a_*)$ | $(b, b_*)$ | $(a, a_*)$, $(c, c_*)$ |
| $(a, d_*)$ | $(a, a_*)$ | $(c, c_*)$ | $(a, a_*)$, $(b, b_*)$, $(c, c_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(c, c_*)$ | $(a, a_*)$ | $(a, a_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(c, c_*)$ | $(b, b_*)$ | $(a, a_*)$, $(c, c_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(c, c_*)$ | $(c, c_*)$ | $(a, a_*)$, $(b, b_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(c, c_*)$, $(d, d_*)$ | $(d, d_*)$ | $(a, a_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(d, d_*)$ | $(a, a_*)$, $(b, b_*)$, $(c, c_*)$ | $(a, a_*)$, $(b, b_*)$, $(c, c_*)$, $(d, d_*)$ |

Table C.2. Observations Identified as Strategic Bound of 2: Constant P2
Shaded is identified in both IDENT and CS; non-shaded is only identified in IDENT.

| P1 | P2 | P3 | P4 |
|---|---|---|---|
| $(a, d_*)$ | $(d, d_*)$ | $(d, d_*)$ | $(a, a_*)$, $(d, d_*)$ |
| $(a, d_*)$ | $(d, d_*)$ | $(a, a_*)$, $(b, b_*)$, $(c, c_*)$ | $(a, a_*)$, $(b, b_*)$, $(c, c_*)$, $(d, d_*)$ |

Table C.3. Observations Identified as Heuristic Bound of 2: Constant P2
For Both IDENT and CS

| | P2 | | P3 | | P4 | |
|---|---|---|---|---|---|---|
| | IDENT | CS | IDENT | CS | IDENT | CS |
| SB Model | (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, d$_*$), (c, a$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$) | (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, d$_*$), (d, a$_*$), (d, c$_*$) | (a, a$_*$),(b, b$_*$),(c, c$_*$),(d, d$_*$) | | (a, a$_*$),(b, b$_*$),(c, c$_*$),(d, d$_*$) | |
| HB Model | (b, a$_*$), (c, a$_*$), (d, a$_*$) | (a, d$_*$), (b, a$_*$), (b, d$_*$), (d, a$_*$) | (a, a$_*$),(b, b$_*$),(c, c$_*$),(d, d$_*$) | | (a, a$_*$),(b, b$_*$),(c, c$_*$),(d, d$_*$) | |

Table C.4. Identified as Strategic/Heuristic Bound of 2: Non-Constant P2

| | P2 | | P3 | | P4 | |
|---|---|---|---|---|---|---|
| | IDENT | CS | IDENT | CS | IDENT | CS |
| SB Model | (a, a$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, d$_*$), (c, a$_*$), (c, c$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$), (d, d$_*$) | (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, c$_*$), (c, d$_*$), (d, a$_*$), (d, c$_*$), (d, d$_*$) | (a, b$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, c$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, d$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$) | (a, b$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, c$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, d$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$) | (a, a$_*$),(b, b$_*$),(c, c$_*$),(d, d$_*$) | |
| HB Model | (b, a$_*$), (c, a$_*$), (d, a$_*$), (d, d$_*$) | (a, d$_*$), (b, a$_*$), (b, d$_*$), (d, a$_*$), (d, d$_*$) | (a, b$_*$), (c, b$_*$),(d, a$_*$),(d, b$_*$) | (a, b$_*$), (b, a$_*$), (c, a$_*$),(c, b$_*$) | (a, a$_*$), (b, b$_*$), (c, c$_*$), (d, d$_*$) | |

Table C.5. Identified as Strategic/Heuristic Bound of 3

| | P2 | | P3 | | P4 | |
|---|---|---|---|---|---|---|
| | IDENT | CS | IDENT | CS | IDENT | CS |
| SB Model | (a, a$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, d$_*$), (c, a$_*$), (c, c$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$), (d, d$_*$) | (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, c$_*$), (c, d$_*$), (d, a$_*$), (d, c$_*$), (d, d$_*$) | (a, a$_*$), (a, b$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, b$_*$), (b, c$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, c$_*$), (c, d$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$), (d, d$_*$) | (a, a$_*$), (a, b$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, b$_*$), (b, c$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, c$_*$), (c, d$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$), (d, d$_*$) | (a, b$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, c$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, d$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$) | (a, b$_*$), (a, c$_*$), (a, d$_*$), (b, a$_*$), (b, c$_*$), (b, d$_*$), (c, a$_*$), (c, b$_*$), (c, d$_*$), (d, a$_*$), (d, b$_*$), (d, c$_*$) |
| HB Model | (b, a$_*$), (c, a$_*$), (d, a$_*$), (d, d$_*$) | (a, d$_*$), (b, a$_*$), (b, d$_*$), (d, a$_*$), (d, d$_*$) | (a, a$_*$), (a, b$_*$), (c, b$_*$),(d, a$_*$),(d, b$_*$) | (a, a$_*$), (a, b$_*$), (b, a$_*$), (c, a$_*$),(c, b$_*$) | (a, c$_*$), (b, c$_*$), (d, b$_*$), (d, c$_*$) | (a, b$_*$),(a, c$_*$), (b, c$_*$), (c, b$_*$),(d, b$_*$), (d, c$_*$) |

Table C.6. Identified as Strategic/Heuristic Bound of 4

## Appendix D Experimental Instructions

**Zoom Meeting: Set-Up** A recurring Zoom meeting was created with the following features. A waiting room was enabled and participants could not join before host. Participants' video was set to 'on,' but participants were muted upon entry. The Zoom chat feature was set to 'chat with host only.' In addition, participants could not rename themselves or provide meeting reactions and non-verbal feedback. Experimenter names were anonymized to "Main Experimenter" and "Experimental Assistant X" (where $X$ took on one of several letters).

**Zoom Meeting: Check-in** Experimental Assistants allowed subjects into the Zoom meeting one at a time. They ensured that subjects had their video on and checked the subjects' names. Then, they changed the subject's name to a pre-specified number and put the subject back into the waiting room to await the start of the experiment. (A Powerpoint slide explained this to the subjects.) Messages went out regularly to subjects in the waiting room, to ensure that they understood that the experiment would soon start and that they would be required to have their video on throughout.

**Zoom Meeting: Start of the Experiment** After check-in was completed, subjects were allowed into the Zoom session. At that time, the Main Experimenter read the preliminary instructions described below. During that time, Experimental Assistants distributed links to the experiment. When all links were distributed, the experiment began. At this time, the Zoom meeting was set to "subjects cannot unmute themselves."

**Preliminary Instructions** The following is the text for the preliminary instructions read at the start of the session.

- Can you hear me? Please nod if you can. Thank you. If you cannot, please check your audio and volume.
- Thank you for agreeing to participate in today's ELFE experiment.
- You will soon receive a link to the experiment in the Zoom chat. It is important that you do not click on the link until I tell you to do so: If you click too early, it will likely take longer for everyone involved.
- Let me begin by telling you some important features about the experiment.
- It is important that you give the experiment your full attention. For the duration of the experiment, please do not use your mobile phone or engage in other activities. In

addition, please keep your video camera on at all times and remain visible at your computer. If your video camera is not on, we will need to remove you from today's session.

- The experiment will begin with a consent form. Please read it carefully and be sure that you understand what you are consenting to. After that, you will receive a series of instructions for the experiment followed by a quiz that is intended to ensure that you have understood the instructions. It is important that you read the instructions carefully so that you can complete the quiz. If you answer the quiz correctly on the first try, you will earn an additional quiz fee described in the experiment. But, even if you don't, it is important that you answer the quiz correctly within three tries. If you do, you will have the opportunity to earn considerably more money during the experiment. The instructions will explain this further. Punchline: You have every incentive to read the instructions carefully.

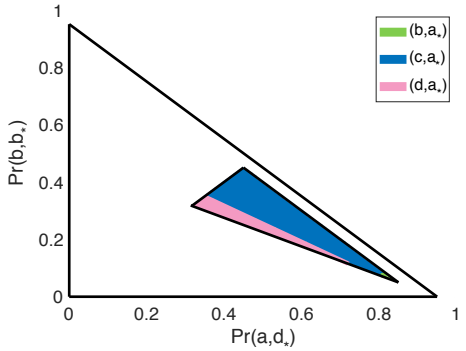
- Once you are admitted to the experiment, you will have 75 minutes to complete it. You will be able to advance through the experiment at your own pace. On each screen, there will be a blue chat icon in the lower left-hand corner. If you have any questions throughout the experiment, you can use that to chat with one of the experimenters.

- After everyone has completed the experiment, your payment information will appear on the screen. At that point, I will make an announcement on Zoom that the experiment is over and you are free to leave the Zoom meeting. If you leave the Zoom meeting before the experiment is over, you may not be paid for your participation in the experiment.

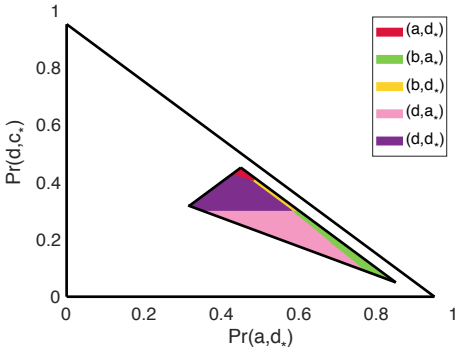
- While you will be able to advance through the experiment at your own pace, the experiment will only "end" after everyone in this session has completed the experiment. In order to receive payment, you will need to wait until the end, after everyone has completed the session, and the Zoom is dismissed. Expect this to take the full session length. Punchline: There is no incentive to finish the experiment quickly. Expect to be here the full session length. Feel free to carefully study the questions, so that your earnings are high.

- Before we begin, are there any questions? If so, please type it into the Zoom chat box and send it to myself—the Main Experimenter.

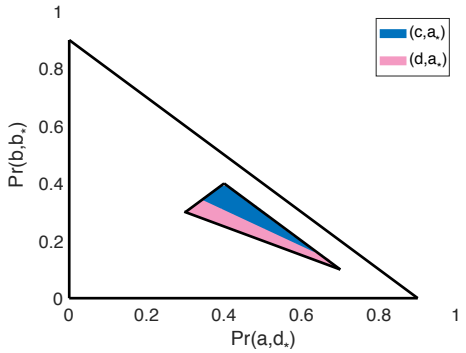
- Please wait quietly until I tell you to start the experiment. Again, please do not click the link until I tell you to do so. Thank you.



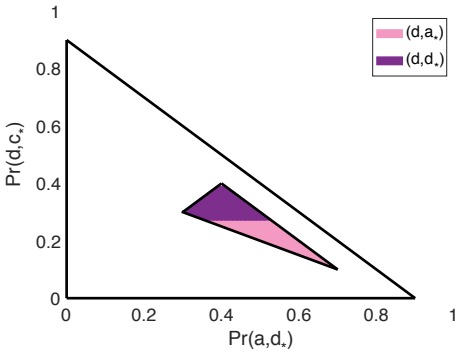

(a) IDENT: $\Pr(\mathrm{d}, \mathrm{c}_*) = .05$



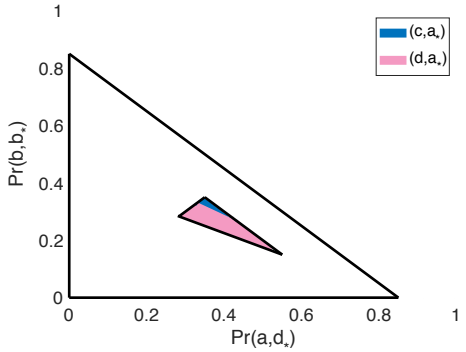

(b) CS: $\Pr(\mathrm{d}, \mathrm{c}_*) = .05$



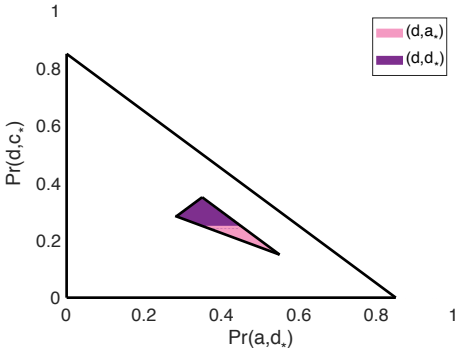

(c) IDENT: $\Pr(\mathrm{d}, \mathrm{c}_*) = .10$



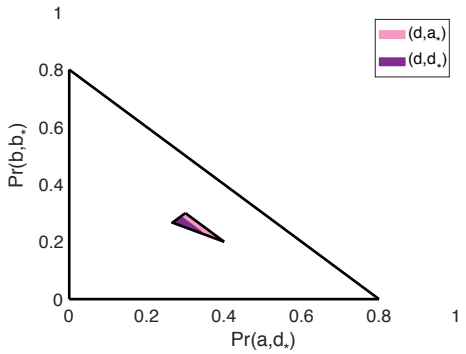

(d) CS: $\Pr(\mathrm{d}, \mathrm{c}_*) = .10$



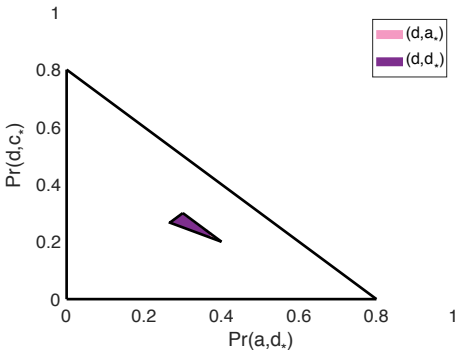

(e) IDENT: $\Pr(\mathrm{d}, \mathrm{c}_*) = .15$




(f) CS: $\Pr(\mathrm{d}, \mathrm{c}_*) = .15$



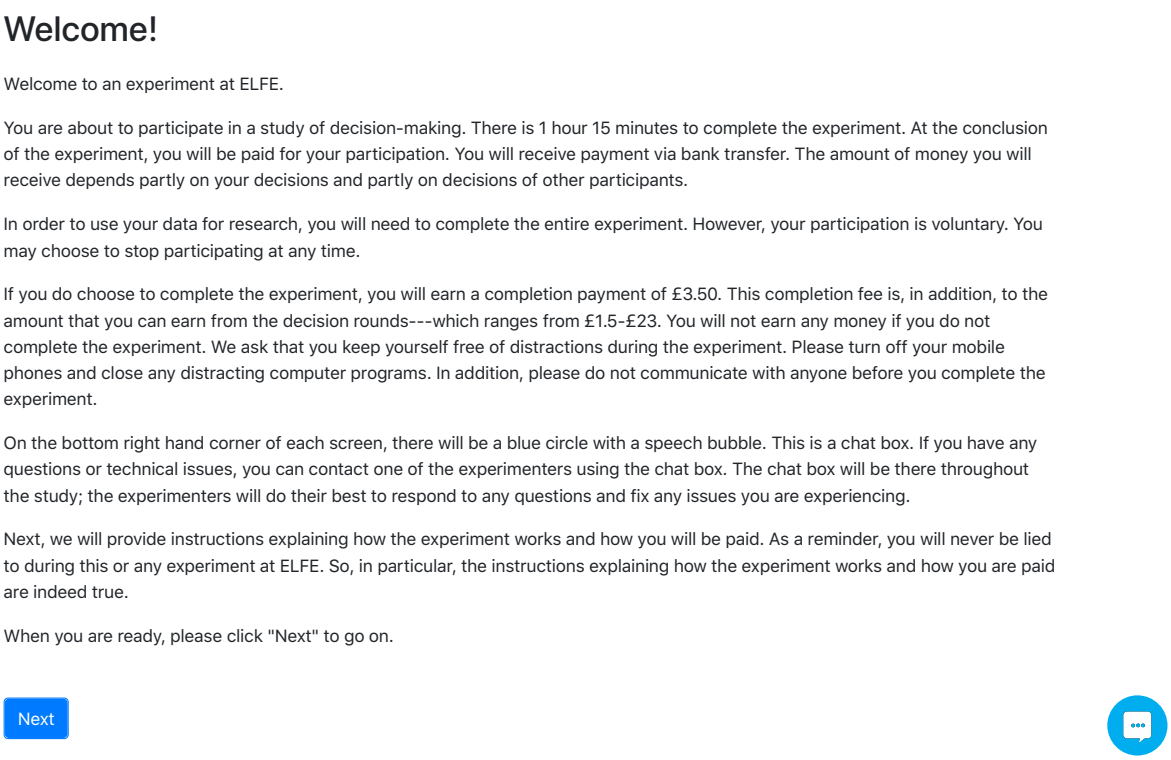

(g) IDENT: $\Pr(\mathrm{d}, \mathrm{c}_*) = .20$




(h) CS: $\Pr(\mathrm{d}, \mathrm{c}_*) = .20$

Figure D.1. P2 Beliefs and Best Responses

**Screenshot of Instructions** Below are screenshots of the instruction. Note, the instructions point to a "highlighting feature" that the subjects can use throughout the game to help them keep track of other players' payoffs. The instructions require the subject to use the highlighting feature before they can move on to a subsequent screen. (For the purposes of producing the screenshots on paper, they are scaled down. Subjects saw larger fonts.)



## Welcome!

Welcome to an experiment at ELFE.

You are about to participate in a study of decision-making. There is 1 hour 15 minutes to complete the experiment. At the conclusion of the experiment, you will be paid for your participation. You will receive payment via bank transfer. The amount of money you will receive depends partly on your decisions and partly on decisions of other participants.

In order to use your data for research, you will need to complete the entire experiment. However, your participation is voluntary. You may choose to stop participating at any time.

If you do choose to complete the experiment, you will earn a completion payment of £3.50. This completion fee is, in addition, to the amount that you can earn from the decision rounds---which ranges from £1.5-£23. You will not earn any money if you do not complete the experiment. We ask that you keep yourself free of distractions during the experiment. Please turn off your mobile phones and close any distracting computer programs. In addition, please do not communicate with anyone before you complete the experiment.

On the bottom right hand corner of each screen, there will be a blue circle with a speech bubble. This is a chat box. If you have any questions or technical issues, you can contact one of the experimenters using the chat box. The chat box will be there throughout the study; the experimenters will do their best to respond to any questions and fix any issues you are experiencing.

Next, we will provide instructions explaining how the experiment works and how you will be paid. As a reminder, you will never be lied to during this or any experiment at ELFE. So, in particular, the instructions explaining how the experiment works and how you are paid are indeed true.

When you are ready, please click "Next" to go on.

Next

# The Experiment

The experiment will begin with a set of instructions detailing what is expected of you. This will include some examples. Following the instructions, you will be given a quiz. After the quiz, you will participate in 8 decision rounds, where you will be paid according to your choices. You will be told when the instructions and quiz have concluded and the decision rounds are about to begin.

**Quiz**

It is important that you understand the instructions before you attempt to take the quiz. If you correctly answer the quiz questions within 3 (or fewer) attempts, you will have the opportunity to earn anywhere between £2-£20 in the decision rounds. If it takes you additional attempts to correctly answer the quiz questions, you will only have the opportunity to earn between £1.5-£1.75 in the decision rounds. In addition, if you correctly answer the quiz questions on the first attempt, you will earn a £3 quiz fee. Thus, it is a good idea to be sure you understand the instructions before attempting the quiz.

As a reminder, you can always ask questions via the chat box. If you get a question incorrect, please feel free to ask the experimenter questions so that you can correctly complete the quiz within 3 attempts.

**Decision Rounds**

You will earn a payment based on the choices you make in the decision rounds. To determine that payment, the computer will randomly select one of the decision rounds. Your payment will be determined by your choices in that round. Importantly, any of the decision rounds can be selected for payment. So, you should treat each decision round like it will be the one that determines your payment.

When you are ready, please click "Next" to go to the instructions for the experiment.

Next

# Instructions

You will be randomly matched with three other participants: Participant 2, Participant 3 and Participant 4. Both you and each of these participants will face 8 decision rounds; in each decision round, you will each make a choice. During the experiment, you will not learn any information about the choices made by other participants and they will not learn any information about the choices you make.

We now describe the decision problem that you will face in each round. You must choose 1 of 4 actions: a, b, c, or d. The three participants you are matched with will also choose 1 of 4 actions. Your earnings will depend on both the action you choose and the action that Participant 2 chooses. The table titled "Your Earnings" (left-hand table below) represents the possible earnings you can receive. Your action determines the row of the table and Participant 2's action determines the column of the table. So you choose amongst actions a, b, c, or d and Participant 2 chooses amongst actions e, f, g, or h. The cell that corresponds to this combination of actions will determine your earnings.

**Your Earnings**

| | | Participant 2's action | | | |
|---|---|---|---|---|---|
| | | e | f | g | h |
| **Your action** | a | 1 | 2 | 3 | 4 |
| | b | 5 | 6 | 7 | 8 |
| | c | 9 | 10 | 11 | 12 |
| | d | 13 | 14 | 15 | 16 |

**Participant 2's Earnings**

| | | Participant 3's action | | | |
|---|---|---|---|---|---|
| | | i | j | k | l |
| Participant 2's action | e | 17 | 18 | 19 | 20 |
| | f | 21 | 22 | 23 | 24 |
| | g | 25 | 26 | 27 | 28 |
| | h | 29 | 30 | 31 | 32 |

**Participant 3's Earnings**

| | | Participant 4's action | | | |
|---|---|---|---|---|---|
| | | m | n | o | p |
| Participant 3's action | i | 33 | 34 | 35 | 36 |
| | j | 37 | 38 | 39 | 40 |
| | k | 41 | 42 | 43 | 44 |
| | l | 45 | 46 | 47 | 48 |

**Participant 4's Earnings**

| | | Your action | | | |
|---|---|---|---|---|---|
| | | a | b | c | d |
| Participant 4's action | m | 49 | 50 | 51 | 52 |
| | n | 53 | 54 | 55 | 56 |
| | o | 57 | 58 | 59 | 60 |
| | p | 61 | 62 | 62 | 64 |

You can easily see the earnings you will get by clicking on rows and columns of the earnings tables. Clicking on a row action will highlight the associated row and show all the possible earnings you can get if you choose that row. Clicking on a column action will highlight the associated column and show all the possible earnings you can get if Participant 2 chooses that column. By highlighting both a row and column, you can see a darker red box that shows what you would get if you choose that row and Participant 2 chooses that column. At any time, you can unhighlight a row or column by clicking on it again. You can also highlight multiple rows or columns at the same time.

Try out clicking the row where you choose a and Participant 2 chooses e. There, you would earn £1, as illustrated by the darker red box. Now try instead clicking the row where you choose b and Participant 2 chooses f. Now you would earn £6.

The earnings tables also show earnings of Participants 2, 3, and 4. Participant 2 can choose one action from e, f, g, h; Participant 3 can choose an action from i, j, k, l; and Participant 4 can choose an action from m, n, o, p. Notice, Participant 2's earnings depend upon the action they choose and the action Participant 3 chooses; Participant 3's earnings depend upon the action they choose and the action Participant 4 chooses; Participant 4's earnings depend upon the action they choose and the action that you choose.

Notice, you can click the rows and columns in each Participant's earning table to see what they will earn. For instance, let's look at Participant 3's earnings table. Try clicking the row where Participant 3 chooses j and the column where Participant 4 chooses m. In that case, Participant 3 will get £37.

Importantly, the earnings tables will differ from round-to-round. You should always look carefully at the earnings tables at the beginning of each round to determine your earnings for that round.

You will be required to spend at least 90 seconds on each round. You may spend more time on each round, if you wish.

The "Next" button will appear after 2 minutes. Please use this time to carefully read the instructions. When you understand these instructions, please click "Next" to go to final remarks on the experiment. You can always come back to this page by clicking the button called "Instructions."

Next

## Instructions: Final Remarks

In each round, you will be randomly matched with three participants. The identity of your randomly matched counterparts will never be revealed. Likewise, your randomly matched counterparts will not know that they are matched with you. During the experiment, you will not learn any information about the choices made by other participants and they will not learn any information about the choices you make.

As a reminder, your earnings will consist of three components. First, you will earn a completion payment of £3.50 for completing the experiment. Second, you will earn a £3 quiz fee, if you answer all of the quiz questions correctly on the first attempt. Third, 1 of the 8 decision rounds will be randomly selected for payment. If you pass the quiz successfully in 3 attempts then this payment will be at least £2 and can be as high as £20. Otherwise, you will play the decision rounds for lower stakes and this payment will be between £1.5 and £1.75. Note, you must complete the entire experiment to earn either of the completion fee, the quiz fee, or the decision rounds payment.

After all participants have completed the experiment, you will see information about your earnings: Specifically, you will see the round chosen for payment, the choice you made in that round, the choices your matched counterparts made in that round, and your total earnings.

We do our best to deliver payments in a timely fashion, but please allow up to 48 hours to receive your payment. If your payment does not arrive within 48 hours, please contact experiments@amandafriedenberg.org.

Please click "Next" when you are ready to take the practice quiz.

Next

Show Instructions

**Screenshot of Quiz** Below is screenshots of the quiz.

## Quiz

**Your Earnings**

| | | Participant 2's action | | | |
|---|---|---|---|---|---|
| | | e | f | g | h |
| **Your action** | a | 45 | 36 | 61 | 43 |
| | b | 7 | 100 | 39 | 19 |
| | c | 87 | 23 | 15 | 73 |
| | d | 64 | 66 | 3 | 38 |

**Participant 2's Earnings**

| | | Participant 3's action | | | |
|---|---|---|---|---|---|
| | | i | j | k | l |
| Participant 2's action | e | 85 | 96 | 38 | 5 |
| | f | 74 | 27 | 9 | 73 |
| | g | 58 | 93 | 65 | 35 |
| | h | 18 | 23 | 19 | 67 |

**Participant 3's Earnings**

| | | Participant 4's action | | | |
|---|---|---|---|---|---|
| | | m | n | o | p |
| Participant 3's action | i | 39 | 81 | 62 | 25 |
| | j | 63 | 75 | 58 | 46 |
| | k | 3 | 82 | 54 | 23 |
| | l | 92 | 39 | 28 | 81 |

**Participant 4's Earnings**

| | | Your action | | | |
|---|---|---|---|---|---|
| | | a | b | c | d |
| Participant 4's action | m | 99 | 81 | 2 | 89 |
| | n | 3 | 99 | 69 | 90 |
| | o | 54 | 7 | 79 | 63 |
| | p | 9 | 94 | 54 | 14 |

Consider the above game.

1. Your earnings depend on your action and the action of which other participant?

- ○ Participant 2
- ○ Participant 3
- ○ Participant 4

2. Participant 3's earnings depend on his/her action and the action of which other participant?

- ○ Your action
- ○ Participant 2
- ○ Participant 4

Suppose you choose d, Participant 2 chooses f, Participant 3 chooses k and Participant 4 chooses m.

3. Please highlight the above choices in the earnings tables. That is, in the table titled "Your Earnings," highlight the row action d and the column action f; in the table titled "Participant 2's Earnings," highlight the row action f and the column action k; in the table titled "Participant 3's Earnings," highlight the row action k and the column action m; in the table titled "Participant 4's Earnings," highlight the row action m and the column action d.

4. What will your earnings be?

5. What will Participant 2's earnings be? (Enter integer):

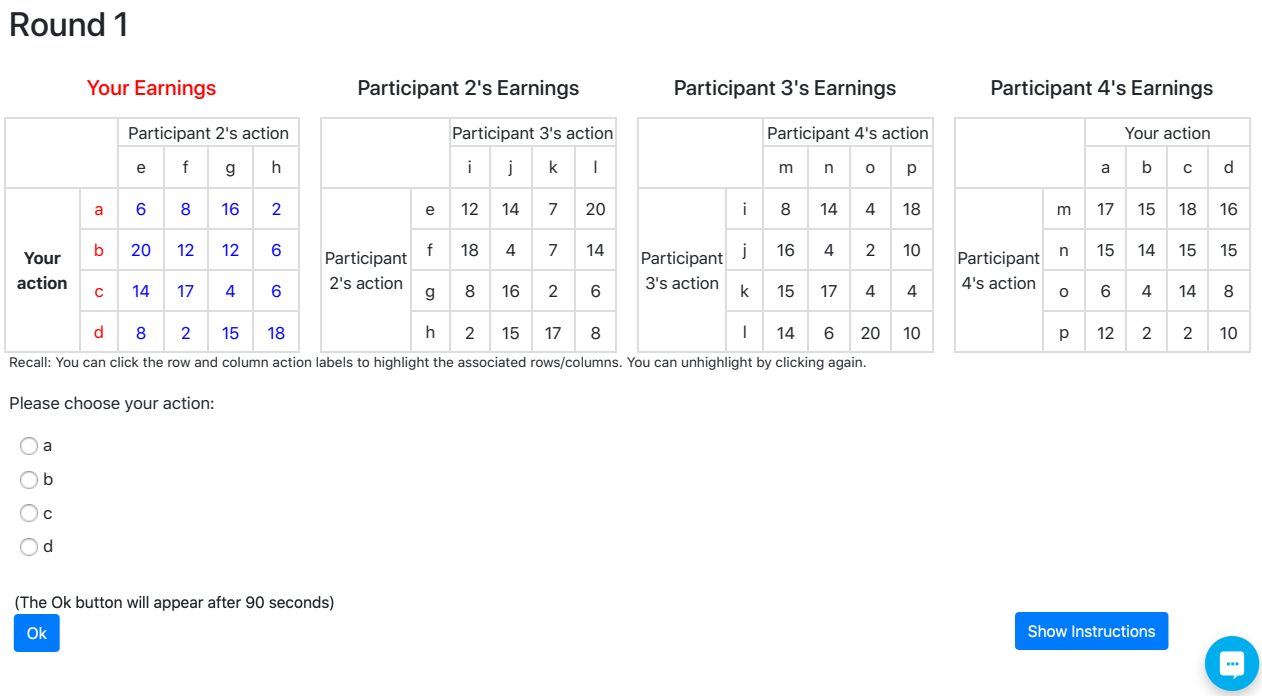
6. What will Participant 3's earnings be? (Enter integer):

7. What will Participant 4's earnings be? (Enter integer):

Please click "Next" when you are ready to submit your answers

Next Show Instructions

**Screenshot of Example Game** Below is a screenshot of Treatment 1, Game $G$, role P4.




# Appendix E Error Models

This appendix introduces two error models. They each capture the idea that the observed distribution of play is generated by subjects who have no gap between their rationality and strategic bounds. Instead, all "non-predicted" behavior is viewed as noise.

Each model features a set of types

$$T = \{r^1, r^2, r^3, r^4\}$$

and a prior on types $\pi \in \Delta(T)$. In both models, type $r^m$ plays the IU strategy in the role of P$i$ for $i \leq m$ and, otherwise, plays a constant strategy. But errors may lead to a different observation. The two models will differ in how errors are generated. The first model, a Random Choice model, naturally captures measurement errors. The second model, a Logistic Choice model, is typically used to model mistakes in decision-making.

**Random Choice Model** The first model is a **Random Choice** (**RC**) **model**, denoted $\mathcal{M}^{\text{RC}} = (T, \pi, \varepsilon)$. In the RC model, the errors are independent of the payoffs and the player role. In each player role, a type follows their set of predicted strategies with probability $1-\varepsilon$ and trembles with probability $\varepsilon$. For example, consider type $r^3$. In the role of P4, the type plays one of the constant strategies with probability $\frac{1-\varepsilon}{4}$ and plays any of the other twelve strategies with probability $\frac{\varepsilon}{12}$. In the role of P3, the subject plays the IU strategy

with probability $1-\varepsilon$ and plays any of the other fifteen strategies with probability $\frac{\varepsilon}{15}$. And similarly for her play in the role of P2 and P1.

With this in mind, write $x_{g,\mathrm{IU}} = (x_{g,\mathrm{IU}}(1), \ldots, x_{g,\mathrm{IU}}(4))$ for the IU strategy profile in the game $g$. Let $\mathbf{1}_{g,\mathrm{IU}}(x,i) = 1$ if $x(i) = x_{g,\mathrm{IU}}(i)$ and $\mathbf{1}_{g,\mathrm{IU}}(x,i) = 0$ otherwise. Likewise, let $\mathbf{1}_{\mathrm{con}}(x,i) = 1$ if $x(i)$ is constant and 0 otherwise. So, $\mathbf{1}_{g,\mathrm{IU}}$ (resp. $\mathbf{1}_{\mathrm{con}}(x,i)$) is an indicator that takes each $(x,i)$ and specifies whether $x$ specifies the IU (resp. constant) strategy profile in the game $g$ for P$i$. Then the probability of observing $x$ in the model $\mathcal{M}^{\mathrm{RC}}$ given a subject of type $r^m$ is

$$\Pr(x,\varepsilon|r^m) = (1-\varepsilon)^{\sum_{i\leq m}\mathbf{1}_{g,\mathrm{IU}}(x,i)}\left(\frac{\varepsilon}{15}\right)^{\sum_{i\leq m}(1-\mathbf{1}_{g,\mathrm{IU}}(x,i))}\left(\frac{1-\varepsilon}{4}\right)^{\sum_{i>m}\mathbf{1}_{\mathrm{con}}(x,i)}\left(\frac{\varepsilon}{12}\right)^{\sum_{i>m}(1-\mathbf{1}_{\mathrm{con}}(x,i))}$$

Then, the likelihood of observing $x$ in the model $\mathcal{M}^{\mathrm{RC}}$ is

$$\mathcal{L}(x \mid \mathcal{M}^{\mathrm{RC}}) = \sum_{r^m\in T}\pi(r^m)\Pr(x,\varepsilon|r^m).$$

**Logistic Choice Model** The second model is a **Logistic Choice** (**LC**) **model**, denoted $\mathcal{M}^{\mathrm{LC}} = (T,\pi,\lambda)$. In the LC model, errors depend on the expected payoffs in a given player role.

With this in mind, fix an action e (resp. e$_*$), write $\mathrm{E}u_i(\mathrm{e}|G,\mathrm{IU})$ (resp. $\mathrm{Eu}_{*,i}(\mathrm{e}_*|G_*,\mathrm{IU})$) for the expected payoff from playing e (resp. e$_*$) in the role P$i$, when P$i$ expects P$(i-1)$ to play the IU strategy. Likewise, write $\mathrm{Eu}_i(\mathrm{e}|G,\mathrm{uni})$ (resp. $\mathrm{Eu}_{*,i}(\mathrm{e}_*|G_*,\mathrm{uni})$) for the expected payoff from playing e (resp. e$_*$) in the role P$i$, when P$i$ expects P$(i-1)$ to play uniformly. Then define *logistic best response functions for Pi* as $\sigma_i$ as a mapping from $\{G,G_*\}\times\{\mathrm{IU},\mathrm{uni}\}\times\mathbb{R}_+$ to distributions on actions so that

$$\sigma_i(G,\mathrm{belief},\lambda)(\mathrm{e}) = \frac{\mathrm{Exp}(\lambda\cdot\mathrm{Eu}_i(\mathrm{e}|G,\mathrm{belief}))}{\sum_{\mathrm{f}\in\{\mathrm{a,b,c,d}\}}\mathrm{Exp}(\lambda\cdot\mathrm{Eu}_i(\mathrm{f}|G,\mathrm{belief}))},$$

and

$$\sigma_i(G_*,\mathrm{belief},\lambda)(\mathrm{e}_*) = \frac{\mathrm{Exp}(\lambda\cdot\mathrm{Eu}_i(\mathrm{e}_*|G_*,\mathrm{belief}))}{\sum_{\mathrm{f}_*\in\{\mathrm{a}_*,\mathrm{b}_*,\mathrm{c}_*,\mathrm{d}_*\}}\mathrm{Exp}(\lambda\cdot\mathrm{Eu}_i(\mathrm{f}_*|G_*,\mathrm{belief}))}.$$

Fix a game $g\in\{G,G_*\}$. A logistic best response function for P$i$, viz. $\sigma_i(g,\mathrm{belief},\lambda)$, specifies a distribution over actions played in the role of P$i$. It assigns this based on the expected payoff from each action, where the expected payoff is either based on a belief that others play IU (i.e., "belief" is IU) or a uniform belief (i.e., "belief" is uni). The action that gives the highest expected payoff is played with the highest probability; the action that gives the lowest expected payoff is played with the lowest probability. These probabilities depend on a precision parameter $\lambda$. Increasing $\lambda$ increases the likelihood of playing the action with the highest expected payoff.

Then the probability of observing $x$ in the model $\mathcal{M}^{\mathrm{LC}}$ given a subject of type $r^m$ is

$$\Pr(x,\lambda|r^m)=\prod_{i=1}^{m}\left(\sigma_i(G,\mathrm{IU},\lambda)(x^G(i))\sigma_i(G_*,\mathrm{IU},\lambda)(x^{G_*}(i))\right)\prod_{i>m}\left(\sigma_i(G,\mathrm{uni},\lambda)(x^G(i))\sigma_i(G_*,\mathrm{uni},\lambda)(x^{G_*}(i))\right)$$

where $x^G(i)$ (resp. $x^{G_*}(i)$) is the action $x(i)$ specifies in $G$ (resp. $G_*$). Then, the likelihood of observing $x$ in the model $\mathcal{M}^{\mathrm{LC}}$ is

$$\mathcal{L}(x\mid\mathcal{M}^{\mathrm{LC}})=\sum_{r^m\in T}\pi(r^m)\Pr(x,\lambda|r^m).$$

## E.1 Best Fitting Error Models

The dataset consists of $N=295$ observations $x_n=(x_n(1),x_n(2),x_n(3),x_n(4))$. Write $\boldsymbol{x}=(x_n)_{n=1}^N$ for this dataset. The aggregate log-likelihood of observing $\boldsymbol{x}$ in $\mathcal{M}$ is

$$\ln\mathcal{L}(\boldsymbol{x}\mid\mathcal{M})=\sum_{n=1}^{N}\ln\mathcal{L}(x_n\mid\mathcal{M}).$$

We use maximum likelihood to find the best-fitting RC and the best-fitting LC models. Table E.1 provides the estimated parameters for the best-fitting models.

| **Random Choice** | | **Logistic Choice** | |
|---|---|---|---|
| $\pi(r^1)$ | .31 | $\pi(r^1)$ | .32 |
| $\pi(r^2)$ | .41 | $\pi(r^2)$ | .31 |
| $\pi(r^3)$ | 0 | $\pi(r^3)$ | .08 |
| $\pi(r^4)$ | 0.28 | $\pi(r^4)$ | .29 |
| $\varepsilon$ | .29 | $\lambda$ | .87 |

Table E.1. Random Choice and Logistic Choice Model Estimates

## E.2 Distribution of Play Generated by Error Models

We used the best-fitting error models to simulate the expected distribution of play. In particular, we ran 1000 simulations of both models in Table E.1.

For each simulated dataset, we assess whether each simulated observation is consistent with our identifying assumptions on strategic behavior. If they are, we assign a strategic bound and a rationality bound. This data is then aggregated into three groups. First,

there are *inconsistent observations*, whose behavior is inconsistent with our identifying assumptions. Second, there are *no-gap observations*, whose identified strategic and rationality bounds coincide. Third there are *gap observations*, whose identified strategic bounds are strictly higher than their identified rationality bounds. We do this for both the identified strategic bounds in the SB model and the identified strategic bounds in the HB model.

Figures E.1 and E.2 give the mean results over the 1000 simulations. Both error models generate far more inconsistent observations than are actually observed in the data. This is particularly striking under the HB Model, where almost 70% of the simulated observations are classified as inconsistent with the HB assumptions for both models. Further, in both error models, gap and no-gap observations are under-represented relative to the data.

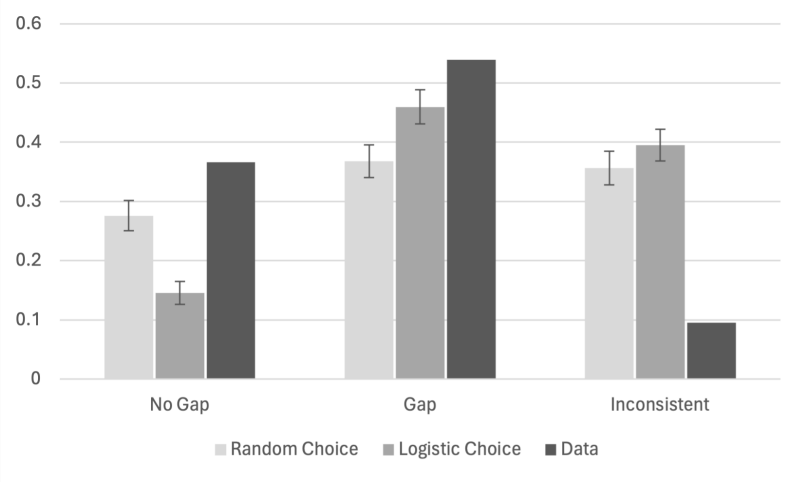



Figure E.1. Identified Gap of Simulated Data under SB

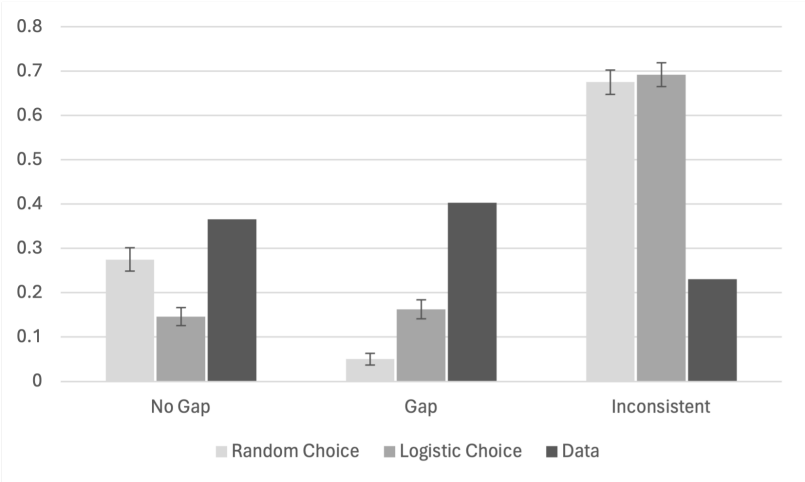



Figure E.2. Identified Gap of Simulated Data under HB

| $\pi(r^1)$ | $\pi(r^2)$ | $\pi(r^3)$ | $\pi(r^4)$ | $\pi(s^2)$ | $\pi(s^3)$ | $\pi(s^4)$ | $\varepsilon$ |
|---|---|---|---|---|---|---|---|
| .17 | .05 | .00 | .17 | .22 | .10 | .28 | .09 |

Table E.2. Random Choice with Gaps

### E.3 Both Gaps and Errors

Consider now a modification of the error models. Now there are two categories of subjects. One category behaves as before: There is some $m$ so that they play the IU strategy in the role P$i$ for $i \leq m$ and, otherwise, plays a constant strategy. The second category behaves as in the HB model. However, errors can lead to observations inconsistent with these assumptions.

So, in particular, now the type set is

$$T = \{r^1, r^2, r^3, r^4, s^2, s^3, s^4\}$$

and the prior is $\pi \in \Delta(T)$. The interpretation of $r^m$ is as on page 19. For each $k \geq 2$, type $s^k$ can play any observation consistent with a heuristic bound of $k$. (This will be given by Tables C.4-C.6.)

**Remark E.1.** Certain observations that are generated by a subject with a rationality bound of 1 can also be generated by a subject with a heuristic bound of 1 or a heuristic bound of 2. In those cases, we view the observation as being generated only by a subject with a rationality bound of 1. We do so to bias the results against the HB model, making it less likely that any such observation would have arisen from HB.

We view errors as generated by a random choice model. We focus on this case because subjects with a heuristic bound of $k \geq 2$ are not easily associated with a belief/an expected payoff, as required by a logistic choice model.[3] Thus, the probability of observing $x$ in the model given a subject of type $r^m$ is as in the RC model. The probability of observing $x$ given a subject of type $s^k$ is computed analogously from Tables C.4-C.6. The likelihood of observing the dataset is computed as earlier. We use maximum likelihood to find the best-fitting model. Table E.2 gives the results.

In the best fitting model, the probability that the data was generated by a subject whose rationality bound and strategic bound coincide (i.e., a subject classified as $r^1, r^2, r^3, r^4$) is .39. The probability it was generated by a heuristic subject (i.e., a subject classified as $s^2, s^3, s^4$) is .60. This fits with the lesson of Table 4.1. Interestingly, the estimated error is 9%. Recall, in Section 6.1, we discussed why we would expect an error rate of 9%.

[3]We could look at a mixed model, where errors of $r^m$ are generated by logistic choice and errors of $s^k$ are generated by random choice. The table is available upon request and the conclusion is the same.